10.2, 10.3, and 13.2 Review

This review sheet discusses, in a very basic way, the key concepts from these sections. This review is not meant to be all inclusive, but hopefully it reminds you of some of the basics. Please notify me if you find any typos in this review.

1. **10.2 Calculus for Parametric Curves:** You need to be able to perform all basic calculus operations directly from a set of parametric equations. In 10.2, we only talked about parametric curves in 2 dimensions. In this case, we started with

$$x = f(t) \quad \text{and} \quad y = g(t).$$

We noted that $\frac{dx}{dt} = f'(t)$ and $\frac{dy}{dt} = g'(t)$, and we used these facts to rewrite formulas from calculus in terms of the parameter t. Here is a summary:

First Derivative: $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Second Derivative: $$\frac{d^2y}{d^2x} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}$$

Integral/Area: $$\int_a^b y\ dx = \int_\alpha^\beta g(t)f'(t)\ dt$$

Arc Length: $$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

Surface Area: $$\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx = \int_\alpha^\beta 2\pi g(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt \qquad \text{AND}$$

$$\int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx = \int_\alpha^\beta 2\pi f(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

You should know how to use these.

2. **10.3 Polar Coordinates:** You should know what polar coordinates are, how to graph equations expressed in polar coordinates, and the basic calculus of polar coordinates.

   (a) *What are polar coordinates?*: The polar coordinate method is another way to describe points in 2 dimensions. It is often useful when describing curves involving circular or elliptic arcs. You should read my supplement to Worksheet 4 (and my lecture notes and the text) to make sure you know how to plot points using the polar coordinate method.

   To go back and forth between polar coordinates and cartesian coordinates you need to the following identities:

$$\begin{array}{ll} x = r\cos(\theta) & y = r\sin(\theta) \\ x^2+y^2 = r^2 & \tan(\theta) = \frac{y}{x} \end{array}$$

   (b) *How to plot polar equations?*: There are two main strategies that we will use to plot polar equations:

      i. STRATEGY I: Rewrite in Cartesian coordinates. To do this you will need to use the four identities given above. That is, you want to get rid of all r and θ variables and end up with only x and y variables using only the four identities given. Sometimes, you will need to be a little clever to accomplish this. Other times it may be nearly impossible, or the resulting x, y equations may not be any simpler (In these cases, you have to use Strategy II).

ii. STRATEGY II: Plot points. Select value of θ and compute values of r (that is, make a table of points). Remember then when you plot these points you must use the polar coordinate method. Typically, we select points like $\theta = 0, \pi/4, \pi/2, \ldots$.

(c) *How do we perform basic calculus with polar equations?* The only calculus we did in this section pertained to finding slopes of tangent lines. That is, finding $\frac{dy}{dx}$. If you are interested in integrals, areas, or arc lengths in polar coordinates you will need to read 10.4 (however, you are not required to know this material for the exam).

Here we start with some equation of the form $r = f(\theta)$ and we derived the expression for $\frac{dy}{dx}$ in terms of this function by using the identities. We arrived at:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

3. **13.2 Derivative and Integrals of Vector Functions:** Now we are embarking on calculus in 3 dimensions. Ultimately, derivatives or integrals of vector functions simply means take the derivative or integrals of three separate functions. What's more difficult is really getting a good sense of what this information gives you.

If you want a preview of one application of calculus in 3 dimensions, you are welcome to read ahead to the Applied Project on pages 880 and 881 of your text. This shows you how to prove that the orbits of planets are ellipses assuming Kepler's laws. It is a small glimpse of the usefulness of calculus in the advanced sciences.

Here are the basic calculus definitions of this section (everything in bold is a vector):

(a) *DERIVATIVES*: If $\mathbf{r}(t) = < f(t), g(t), h(t) >$, then

$$\begin{aligned} \mathbf{r}'(t) &= < f'(t), g'(t), h'(t) > \\ &= \text{`a vector tangent to the space curve.'} \end{aligned}$$

Sometimes, we want the tangent vector to have length one. In which case, we compute

$$\frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \text{`the tangent vector with length 1'}.$$

Make sure that you understand the vector derivative rules on page 859 of your text. In particular, notice that their are three types of product rules (scalar function product rule, dot product rule, and cross product rule). These three all have a similar format put with different product operations.

The second derivative is defined by:

$$\mathbf{r}''(t) = < f''(t), g''(t), h''(t) > .$$

And similarly for higher derivatives.

(b) *INTEGRALS*: If $\mathbf{r}(t) = < f(t), g(t), h(t) >$, then

$$\int_a^b \mathbf{r}(t)\ dt = < \int_a^b f(t)\ dt, \int_a^b g(t)\ dt, \int_a^b h(t)\ dt >$$

So you can compute definite integrals using the fundamental theorem of calculus. That is, if $\mathbf{R}'(t) = \mathbf{r}(t)$ (that is $\mathbf{R}(t)$ is an antiderivative vector function), then

$$\int_a^b \mathbf{r}(t)\ dt = \mathbf{R}(b) - \mathbf{R}(a).$$

The indefinite integral is defined as in Math 125. That is,

$$\int \mathbf{r}(t)\ dt = \text{`the general antiderivative vector function'}$$

Your general antiderivative will have a '$+\mathbf{c} =< c_1, c_2, c_3 >$' term.

4. **Aside about Parabolas, Ellipses, and Hyperbolas**: In class, we briefly mentioned this graphs because they tend to pop up when we are drawing traces for 3D graphs. If you are having a tough time remembering the properties of these functions. I suggest you take a look at section 10.5 of your text.

   I have given pictures of an ellipse (at the left) and hyperbola (at the right) below.

   The ellipse is given the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If $a = b = r$, then we get the equation for the circle. You can always quickly plot the points when $(0, b)$, $(0, -b)$, $(a, 0)$, and $(-a, 0)$ to get a quick idea of where your ellipse is.

   The hyperbola is given the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$). The first case is pictured below (the second similar, but in the vertical direction). The graph goes through the x-axis at the points $(a, 0)$ and $(-a, 0)$. The graph approaches the lines $y = (b/a)x$ and $y = -(b/a)x$ as they are drawn out indefinitely.