Facts and Definitions about Three-Dimensional Curves

We will write a three-dimensional parametric curve in either of the equivalent forms $x = f(t), y = g(t), z = h(t)$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. Below we discuss such curves.

1. *Visualization*: Given a parametric curve in three-dimensions. We can try to visualizing the motion using the following tools
   - Eliminate the parameter to get equations relating x, y, and z. Then try to visualize the resulting surface over which the motion is occurring.
   - Plot points by choosing values of t and plotting (x, y, z).
   - Use the tools and measures below to discuss the motion of a curve at a point.
2. *Derivatives and Integrals*:
   - $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle =$ 'a vector tangent to the curve at t' = velocity vector.
   - $\mathbf{r}''(t) = \langle f''(t), g''(t), h''(t)\rangle =$ acceleration vector.
   - $\displaystyle\int \mathbf{r}(t)dt = \left\langle \int f(t)dt, \int g(t)dt, \int h(t)dt \right\rangle.$
3. *Measurements on the Curve*:
   - Arc Length $= \displaystyle\int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$
   - Curvature $= \kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$. You can calculate curvature for a 2D curve as well by making the third component zero. For a function of the form $y = f(x)$ in 2D, the formula for curvature becomes $\kappa = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$.
   - The tangential and normal components of acceleration will be covered in 13.4 and are given by: $a_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$ (tangential component) and $a_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$ (normal component)
4. *Normal Vectors*: We define $\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) =$ the unit tangent. And from it we get the following

   $\mathbf{T}'(t) =$ 'a normal vector (a vector orthogonal to $\mathbf{T}(t)$'

   $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} =$ 'the principal unit normal'

   $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) =$ 'the binormal vector (orthogonal to the tangent and unit normal)'
5. *Related Planes and Lines*
   - The *tangent line* to a curve at a given point can be given by using the $\mathbf{r}'(t)$ as the direction vector in the equations for the line.
   - The *normal plane* to a curve at a given point can be given by using $\mathbf{r}'(t)$ as the normal vector in the equation for a plane.
   - The *osculating plane* to a curve at a given point can be given by using $\mathbf{B}(t)$ as the normal vector in the equation for a plane.