Worksheet 3 Solutions

1. *Surfaces:* For the surface $x + y^2 - z^2 = 4$.

   - If $x = k$ is fixed, then the traces are **hyperbolas**.
   - If $y = k$ is fixed, then the traces are **parabolas**.
   - If $z = k$ is fixed, then the traces are **parabolas**.
   - This shape is called a **hyperbolic paraboloid**.

2. *Basic Parametric:* For $x = t$, $y = t\sin(\pi t)$, $z = t\cos(\pi t)$, describe the surface of motion.
   Answer: Since $x = t$, we can the motion occurs on the intersection of the surfaces $y = x\sin(\pi x)$ and $z = x\cos(\pi x)$. Squaring and summing y and z gives $y^2 + z^2 = x^2\sin^2(\pi x) + x^2\cos^2(\pi x) = x^2$. Thus, the motion occurs on the surfaces $y^2 + z^2 = x^2$. This surface is called a **cone**.

3. *Basic Parametric:* Assume $\mathbf{r}_1(t) = \langle t, 5t, t^2\rangle$ and $\mathbf{r}_2(t) = \langle 5 - t, 7t + 1, t^3 + 1\rangle$.

   - Find all points at which their **paths** intersect.
     Answer: This question is asking if the same (x, y, z) points ever occur (not necessarily with the same parameter choice). So we need to use different symbols for each parameter and then set the x, y and z equations equal.

     $$\begin{array}{lll} (1) & t & = 5 - u \\ (2) & 5t & = 7u + 1 \\ (3) & t^2 & = u^3 + 1 \end{array}$$

     Combining conditions (1) and (2) gives $5(5 - u) = 7u + 1$, which simplifies to $24 = 12u$. Thus, the only way x and y coordinates can be the same is if $u = 2$ and $t = 5 - 2 = 3$. (Note that you can check your work, $u = 2$ and $t = 3$ both give $x = 3$ and $y = 15$).

     And the last step is to see if the z coordinates are the same for these parameter values. For these values, we get $t^2 = 9$ and $u^3 + 1 = 9$, so YES, the curves do intersect.

     The one point of intersection of the two curves if $(x, y, z) = (3, 15, 9)$.

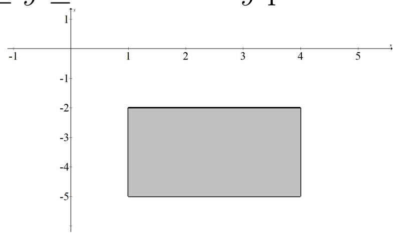
   - Do the object every **collide**? (If so, find the time when they collide. If not, explain why.)
     Answer: If you use the given parameterizations and assume that the parameters are actually time, then this question is asking if the two objects are ever at the same (x, y, z) point at the exact same time t. No additional work is needed to solve this problem. We just saw in the last part that the only way the (x, y, z) point can be the same is when the first time is $t = 3$ and when the second time is $t = 2$. So the answer is NO, the objects do not collide.

4. *Parametric Calculus:* For the curve given by $\mathbf{r}(t) = \langle t^2 + 1, t^3, 1 - 5t\rangle$.

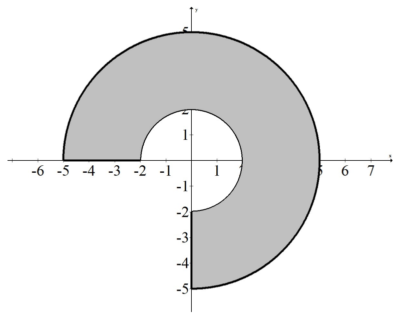
   - *Answer:* $\mathbf{r}'(t) = \langle 2t, 3t^2, -5\rangle$ = the derivative vector (a tangent vector).
   - *Answer:* $\mathbf{r}'(1) = \langle 2, 3, -5\rangle$ and $\mathbf{T}(1) = \frac{1}{\sqrt{4+9+25}}\langle 2, 3, -5\rangle = \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, -\frac{5}{\sqrt{38}}\right\rangle$.
   - *Answer:* $x = 2 + 2t$, $y = 1 + 3t$, $z = -4 - 5t$ because $\mathbf{r}(1) = \langle 2, 1, -4\rangle$ and $\mathbf{r}'(1) = \langle 2, 3, -5\rangle$.

An introduction to Polar Coordinates:

1. The region $1 \leq x \leq 4$ and $-5 \leq y \leq -2$ in the xy-plane:

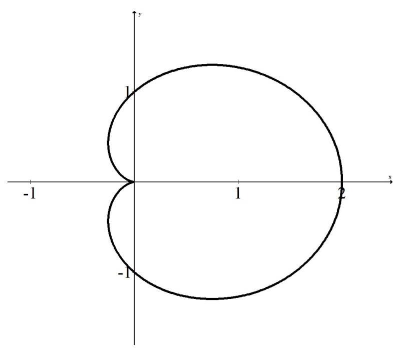


2. Sketch the region in the xy-plane given by all polar points (r, θ) such that $-\frac{\pi}{2} \leq \theta \leq \pi$ and $2 \leq r \leq 5$.

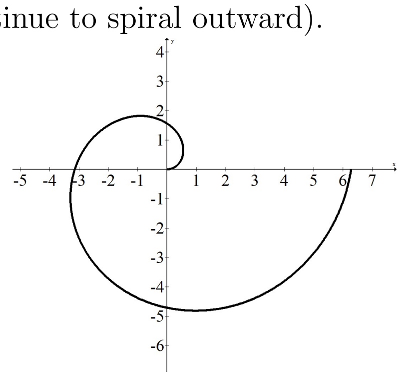


3. Let's plot a couple polar curves.

   - $r = 1 + \cos(\theta)$



   - $r = \theta$ (my answer only draws the graph for the range $\theta = 0$ to $\theta = 2\pi$, if we went beyond this the graph would continue to spiral outward).



4. In 10.2 you learned that if x and y are given in terms of a parameter, then you can find the slope directly from the parameter by using $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. You can use this to find slopes directly from polar curves. The general presentation would be: Suppose $r = f(\theta)$. From our known connections $x = f(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta)$. Note, you can find $dx/d\theta$ using the product rule (same for $dy/d\theta$). Thus, to find the slope you can use

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

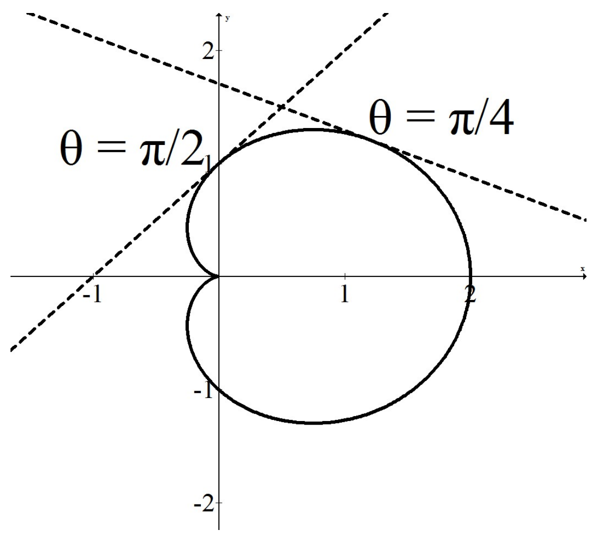
For the cardiod $r = 1 + \cos(\theta)$, use the fact above to find the formula for dy/dx. What is the slope when $\theta = \pi/2$? How about when $\theta = \pi/4$?

Answer: Note that $r = 1 + \cos(\theta)$ and $\frac{dr}{d\theta} = -\sin(\theta)$, so

$$\frac{dy}{dx} = \frac{(-\sin(\theta))\sin(\theta) + (1+\cos(\theta))\cos(\theta)}{(-\sin(\theta))\cos(\theta) - (1+\cos(\theta))\sin(\theta)}$$

At $t = \pi/2$, we get $\frac{dy}{dx} = \frac{(-1)(1)+(1+0)(0)}{(-1)(0)-(1+0)(1)} = \frac{-1}{-1} = 1$. So on the graph at $\theta = \pi/2$ the slope is 1. See below.

At $t = \pi/4$, we get $\frac{dy}{dx} = \frac{(-\sqrt{2}/2)(\sqrt{2}/2)+(1+\sqrt{2}/2)(\sqrt{2}/2)}{(-\sqrt{2}/2)(\sqrt{2}/2)-(1+\sqrt{2}/2)(\sqrt{2}/2)} = \frac{-1/2+(\sqrt{2}/2+1/2)}{-1/2-(\sqrt{2}/2+1/2)} = \frac{\sqrt{2}/2}{-1-\sqrt{2}/2} \approx -0.414$. See below.



Note: You should know how to find the equations for those tangent lines drawn above.