12.5 Review

In 12.5, we learn about lines and planes.

1. **Lines:** Let (x_0, y_0, z_0) be a point on a line and let $\langle a, b, c\rangle$ be any vector that is parallel to the line, then a parametric equation for the line is given by

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

2. **Planes:** Let (x_0, y_0, z_0) be a point on a plane and let $\langle a, b, c\rangle$ be any vectors that is orthogonal to the plane (called a normal vector), then the equation for the plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

3. Stategies to find the equation of a line:
   (a) First read the given information carefully. If you already have a point and a direction, then you are done.
   (b) If the information is not immediately given, then find **two points**: $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$. Then we get a direction vector by subtracting $\overrightarrow{AB} = \langle a, b, c\rangle = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$.

4. Strategies to find the equation of a plane
   (a) First read the given information carefully. If you already have a point and a normal, then you are done.
   (b) If the information is not immediately given, then find **three points** on the plane or, equivalently, find **two vectors** parallel to the plane:
      - If the three points are $A(x_0, y_0, z_0)$, $B(x_1, y_1, z_1)$, and $C(x_2, y_2, z_2)$, then get two vectors parallel to the plane by subtracting:
        $\overrightarrow{AB} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$ and $\overrightarrow{AC} = \langle x_2 - x_0, y_2 - y_0, z_2 - z_0\rangle$.
      - Now we get the normal by taking the cross product of the two vectors $\overrightarrow{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.

5. Finding if two lines intersect:
   Remember if you are trying to find the intersection of two lines (or any two curves given parametrically), then you need to use different parameters. Here are two examples:

   (a) Do the two lines $x = 1 + 2t$, $y = 5 - t$, $z = 3t$ and $x = 4 - u$, $y = 2 + u$, $z = 6 - 2u$ intersect?
   Solution:

   - Note that I already labeled the parameters differently. Now solve the system

   $$\begin{array}{lrl} (1) & 1+2t & = 4-u \\ (2) & 5-t & = 2+u \\ (3) & 3t & = 6-2u \end{array}$$

   - Rearranging (1) gives $u = 3 - 2t$.
   - Substituting (1) into (2) give the combined condition: $5 - t = 2 + (3 - 2t)$. We solve for t to get $t = 0$.
   - Going back to (1) (or (2)) we get $u = 3$. So the only way (1) and (2) can both be true is if $t = 0$ and $u = 3$.
   - Now we check the z-coordinate equation for these values: $3t = 0$ and $6 - 2u = 0$, so yes! There is a point of intersection.
   - The point is $(x, y, z) = (1, 5, 0)$.

   (b) Do the two lines $x = t$, $y = 4 - t$, $z = 8 + t$ and $x = 2u$, $y = 1 - u$, $z = 1 + 3u$ intersect?
   Solution:

   - Note that I already labeled the parameters differently. Now solve the system

   $$\begin{array}{lrl} (1) & t & = 2u \\ (2) & 7-t & = 1-u \\ (3) & 8+t & = 1+3u \end{array}$$

   - Rearranging (1) gives $t = 2u$.
   - Substituting (1) into (2) give the combined condition: $7 - 2u = 1 - u$. We solve for u to get $u = 6$.
   - Going back to (1) (or (2)) we get $t = 12$. So the only way (1) and (2) can both be true is if $t = 12$ and $u = 6$.
   - Now we check the z-coordinate equation for these values: $8 + t = 20$ and $1 + 3u = 19$, so no! There is a no point of intersection.

6. Finding two points on the intersection of two planes:
Two different nonparallel planes intersect to form a line. One strategy to find the equation for that line is to find two points on the line (there are infinitely many points to choose from). Here is an examples of how to find two points on a line:
Find two points on the intersection of the planes $x - y + z = 10$ and $2x + y - 2z = 5$
Solution:

   (a) Combine the equations (by adding or subtracting or substituting). For this example, I'll add the two equations to get $3x - z = 15$.

   (b) Now select ANY value you want for x and compute z and y (or select ANY value you want for z and compute x and y). I will give several examples of points found in this way:

      - If $x = 0$, then you get $z = -15$. Using either of the original equations you can then find y: $y = x + z - 10 = -25$. Thus, $(0, -25, -15)$ is a point on both planes.
      - If $z = 0$, then you get $x = 5$. So $y = x + z - 10 = -5$. Thus, $(5, -5, 0)$ is a point on both planes.
      - If $x = 1$, then you get $z = -12$. So $y = -21$. Thus, $(1, -21, -12)$ is another point on both planes.
      - If $z = 6$, then you get $x = 7$. So $y = 3$. Thus, $(7, 3, 6)$ is another point on both planes.

   And you could continue doing this on and on if you wanted to find more points. For our purposes (finding the equation for the line), we only need to find two points.