## 14.4 and 14.7 Review

This review sheet discusses, in a very basic way, the key concepts from these sections. This review is not meant to be all inclusive, but hopefully it reminds you of some of the basics. Please notify me if you find any typos in this review.

1. **14.4 Tangent Planes**: Know how to find a tangent plane and understand it's basic uses in linear approximation and differentials.

   (a) The **Tangent Plane Equation** for the function $z = f(x, y)$ where $(x_0, y_0, z_0)$ is given by

   $$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

   So you have to first find the partial derivatives and then plug in the values of $(x_0, y_0)$.

   (b) Since $z_0 = f(x_0, y_0)$ and $z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$, we can rewrite the tangent plane as

   $$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

   which we call the **tangent plane approximation** or the **linear approximation** of $f(x, y)$ based at $(x_0, y_0)$.
   If $(x, y)$ is "near" $(x_0, y_0)$, then

   $$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

   **Here is a linear approximation example**:
   Find the linear approximation for $f(x, y) = x\cos(y + 2x) + y^2$ based at $(-1, 2)$ and use it to approximate the value of $(-0.9, 2.05)$.
   ANSWER:

   i. First we find the tangent plane by computing the partial derivatives and get the $z$ value:

   $$f_x(x, y) = \cos(y + 2x) - 2x\sin(y + 2x) \quad f_y(x, y) = -x\sin(y + 2x) + 2y$$
   $$f_x(-1, 2) = \cos(0) - 2(-1)\sin(0) = 1 \quad f_y(-1, 2) = -(-1)\sin(0) + 4 = 4$$
   $$z_0 = f(-1, 2) = -\cos(0) + 4 = 3$$

   So the tangent line approximation is:

   $$z = 3 + 1(x - (-1)) + 4(y - 2) = 3 + (x + 1) + 4y - 8 = x + 4y - 4.$$

   ii. Now we can approximate the value of $f(-0.9, 2.05)$ by using the height on the tangent plane:
   $$z = (-0.9) + 4(2.05) - 4 = 3.3.$$

   (Note that this is very close to the actual value $f(-0.9, 2.05) = 3.33047882$.)

   (c) The **total differential**, $dz$, expresses the tangent plane in terms of the amount of change in $x$, $y$, and $z$. The actual changes measured from the graph of the function $f(x, y)$ are called $\delta x$, $\delta y$, and $\delta z$. The changes measured on the tangent plane are called $dx$, $dy$ and $dz$. You our problems $\delta x = dx$ and $\delta y = dy$. Rewriting the tangent plane equation in terms of differentials we obtain:

   $$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

   Note: As with the tangent line approximation, this only gives an estimation of the actual change in $z$.

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**Here is a differentials example**:

The volume of a cylinder is given by $V(r, h) = \pi r^2 h$. If you manufacture a cylinder that has radius of 3 inches with maximum error of 0.1 inches and height of 8 inches with maximum error of 0.2 inches, given an estimation for maximum error for the volume using differentials.

ANSWER:

i. We want to compute $dV$ when $r = 3$, $dr = 0.1$, $h = 8$, and $dh = 0.2$.

ii. First we find the total differential by computing the partial derivatives:

$$V_r(r, h) = 2\pi r h \qquad\qquad V_h(r, h) = \pi r^2$$
$$V_r(3, 8) = 2\pi(3)(8) = 48\pi \quad V_h(3, 8) = \pi(3)^2 = 9\pi$$

Note that the total differential is

$$dV = 2\pi r h \, dr + pir^2 \, dh.$$

iii. Now we can approximate error to get

$$dV = 48\pi(0.1) + 9\pi(0.2) = 6.6\pi \approx 20.7345.$$

So the volume can be off by about 20.7 in$^3$.

(Aside: Note at $r = 3$, $h = 8$ the actual volume is $72\pi \approx 226.19467$ and at $r = 3.1$, $h = 8.2$ the actual volume is $78.802\pi \approx 247.5638$. Thus, the actual worst error is $247.5638 - 226.19467 \approx 21.3691$.

2. **14.7 Maximum and Minimum Value**: Be able to find critical points and classify them. Also be able to find absolute maxima and minima.

(a) To find the **critical points**, set $f_x(x, y)$ and $f_y(x, y)$ both equal to zero and solve them simulaneously. Any points where $f_x$ or $f_y$ is undefined is also called a critical point.

**Here is a finding critical points example**:

Find the critical points for $f(x, y) = x^2 - 8y^3 + 4xy + 1$.

ANSWER:

i. Find the partial derivatives and set them equal to zero:

$$\begin{array}{ll}
f_x(x, y) = 2x + 4y = 0 & f_y(x, y) = -24y^2 + 4x = 0 \\
\Rightarrow x = -2y & \Rightarrow x = 6y^2 \\
\text{Now combine to get } \Rightarrow & -2y = 6y^2 \\
& 0 = 6y^2 + 2y \\
& 0 = 2y(3y + 1) \\
& y = 0 \text{ or } y = -1/3 \\
\text{go back to get corresponding } x \text{ values} & x = 0 \text{ or } x = 2/3
\end{array}$$

The critical points are $(x, y) = (0, 0)$ and $(x, y) = (2/3, -1/3)$.

(b) Be able to use the **second derivative test** to classify whether the critical point gives a local maximum, local minimum, or a saddle point.

Let $(a, b)$ be a critical point and define

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2,$$

i. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

ii. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

iii. If $D < 0$, then $(a, b)$ is a saddle point.

iv. If $D = 0$, then the test is inconclusive and you will need to use other methods to classify the point.

**Here is a second derivative test example**:

Classify the critical points for $f(x,y) = x^2 - 8y^3 + 4xy + 1$.

ANSWER:

  i. In the last example we found that the critical points are $(0,0)$ and $(2/3, -1/3)$.
  ii. We need to compute the second partial derivatives to find $D$.

$$f_{xx}(x,y) = 2 \quad f_{yy} = -48y \quad f_{xy} = 4$$

  Thus, we have
$$D(x,y) = 2(-48y) - 4^2 = -96y - 16.$$

  iii. The point $(0,0)$ gives $D = -96(0) - 16 = -16$. So $D < 0$, which means that $(0,0)$ is a saddle point.
  iv. The point $(2/3, -1/3)$ gives $D = -96(-1/3) - 16 = 16$. So $D > 0$. So we need to check $f_{xx}(2/3, -1/3) = 2 > 0$, which means that $f(2/3, -1/3)$ is a local minimum. (or we could say that a local minimum occurs at $(2/3, -1/3)$).

(c) All **Absolute Maximum and Minimum Values** occur either at a critical point or on the boundary of the given region. So to find the absolute maximum and minimum values:

  i. Find the critical values in the region and plug them into $f(x,y)$.
  ii. Find formulas for each part of boundary and find values of $f(x,y)$ at these boundary points.
  iii. Absolute max = biggest output, Absolute min = smallest output

The hardest part is often checking the boundaries. These are long problems because they have lots of steps.

**Here is an absolute maximum example**:

Find the absolute maximum and absolute minimum of $f(x,y) = \frac{1}{3}x^3 + 3y^2 - x$ over the rectangular region $R = \{(x,y)|0 \le x \le 3, -1 \le y \le 1\}$.

  i. The critical values occur when $x^2 - 1 = 0$ and $6y = 0$. This only happens when $(x,y) = (-1, 0)$ and $(x,y) = (1,0)$. The point $(1,0)$ is the only critical point in the region. The height of the function at this point is $f(1,0) = -2/3$.
  ii. Now we have to consider all four sides of the rectangle (you should draw it):
    A. $y = -1$ and $0 \le x \le 3$: With $y = -1$, the function becomes $f(x,-1) = \frac{1}{3}x^3 + 3 - x$
    B. $x = 3$ and $-1 \le y \le 1$: With $x = 3$, the function becomes $f(3,y) = 6 + 3y^2$
    C. $y = 1$ and $0 \le x \le 3$: With $y = 1$, the function becomes $f(x,1) = \frac{1}{3}x^3 + 3 - x$
    D. $x = 0$ and $-1 \le y \le 1$: With $x = 0$, the function becomes $f(0,y) = 3y^2$
    Now let's maximize over each side:
    A. $y = -1$ and $0 \le x \le 3$: With $y = -1$, the function becomes $f(x,-1) = \frac{1}{3}x^3 + 3 - x$. The absolute max/min values of this one variable function can be found with Calculus I techniques: Namely, (i) find the critical values and plug them into the function (ii) plug the endpoints into the function (iii) the biggest output is the abs. max, smallest output is the abs. min.
    (i) **Critical Numbers**

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + 3 - x\right) = x^2 - 1 = 0$$

    . Critical Numbers: $x = \pm 1$. Only $x = 1$ is in the domain of interest. So we have

$$f(1,-1) = 7/3 = 2.\bar{3}.$$

    (ii) **Endpoints**
$$f(0,-1) = 3 \qquad f(3,-1) = 9.$$

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(ii) **Absolute Max/Min Over This Side**
So the absolute max and absolute min on this side are 9 and 7/3 (respectively).
And they occur at $(3, -1)$ and $(1, -1)$.

B. $x = 3$ and $-1 \leq y \leq 1$: With $x = 3$, the function becomes $f(3, y) = 6 + 3y^2$.
   (i) **Critical Numbers**
$$\frac{d}{dy}\left(6 + 3y^2\right) = 2y = 0.$$

Critical Numbers: $y = 0$. So we have

$$f(3, 0) = 6.$$

   (ii) **Endpoints**
$$f(3, -1) = 9 \qquad f(3, 1) = 9.$$

   (ii) **Absolute Max/Min Over This Side**
So the absolute max and absolute min on this side are 9 and 6 (respectively). And
they occur at $(3, \pm 1)$ and $(3, 0)$.

C. $y = 1$ and $0 \leq x \leq 3$: With $y = 1$, the function becomes $f(x, 1) = \frac{1}{3}x^3 + 3 - x$.
   (i) **Critical Numbers**

$$\frac{d}{dy}\left(\frac{1}{3}x^3 + 3 - x\right) = x^2 - 1 = 0.$$

Critical Numbers: $x = \pm 1$. Only $x = 1$ is in the domain of interest. So we have

$$f(1, 1) = 7/3 = 2.\overline{3}.$$

   (ii) **Endpoints**
$$f(0, 1) = 3 \qquad f(3, 1) = 9.$$

   (ii) **Absolute Max/Min Over This Side**
So the absolute max and absolute min on this side are 9 and 7/3 (respectively).
And they occur at $(3, 1)$ and $(1, 1)$.
NOTE: This is identical to the $y = -1$ case! So this was redundant and I could
have skipped doing this side (and just used the analysis from the $y = -1$ side).

D. $x = 0$ and $-1 \leq y \leq 1$: With $x = 0$, the function becomes $f(0, y) = 3y^2$.
   (i) **Critical Numbers**
$$\frac{d}{dy}\left(3y^2\right) = 6y = 0.$$

Critical Numbers: $y = 0$. So we have

$$f(0, 0) = 0.$$

   (ii) **Endpoints**
$$f(0, -1) = 3 \qquad f(0, 1) = 3.$$

   (ii) **Absolute Max/Min Over This Side**
So the absolute max and absolute min on this side are 3 and 0 (respectively). And
they occur at $(0, -1)$ and $(0, 1)$ (max occurs twice) and $(0, 0)$ for min.

Thus, all together over the entire region, we found

$$\text{Abs. Max} = f(3, 1) = 9 \qquad \text{Abs. Min} = f(1, 0) = -2/3$$