1 $f(x)=\ln(\ln(x)),\quad b=e \quad\rightarrow\quad f(e)=\ln(\ln(e))=\ln(1)=0$

$f'(x)=\frac{1}{\ln(x)x}=(x\ln(x))^{-1} \quad\rightarrow\quad f'(e)=\frac{1}{e\ln(e)}=\frac{1}{e}$

$f''(x)=-(x\ln(x))^{-2}\left(\ln(x)+x\cdot\frac{1}{x}\right) \quad\rightarrow\quad f''(e)=\frac{-(\ln(e)+1)}{(e\ln(e))^2}=-\frac{2}{e^2}$

$$\boxed{\begin{aligned}T_2(x)&=0+\frac{1}{e}(x-e)+\frac{1}{2!}\frac{-2}{e^2}(x-e)^2\\ T_2(x)&=\frac{1}{e}(x-e)-\frac{1}{e^2}(x-e)^2\end{aligned}}$$

2 $f(x)=\sin\left(\frac{\pi x}{6}\right)\quad b=1 \quad\rightarrow\quad f(1)=\sin(\pi/6)=1/2$

$f'(x)=\pi/6\cos(\pi x/6) \quad\rightarrow\quad f'(1)=\pi/6\cos(\pi/6)=\frac{\pi\sqrt{3}}{12}$

$f''(x)=-\pi^2/6^2\sin(\pi x/6) \quad\rightarrow\quad f''(1)=-\pi^2/36\ \ 1/2=-\pi^2/72$

(a) $$\boxed{\begin{aligned}T_2(x)&=\frac{1}{2}+\frac{\pi\sqrt{3}}{12}(x-1)-\frac{1}{2!}\frac{\pi^2}{72}(x-1)^2\\ T_2(x)&=\frac{1}{2}+\frac{\pi\sqrt{3}}{12}(x-1)-\frac{\pi^2}{144}(x-1)^2\end{aligned}}$$

(b) Interval: $I=[1,1.1]$

$\boxed{\text{STEP 1}}$ Next Derivative: $f'''(x)=-\frac{\pi^3}{6^3}\cos(\pi x/6)$

$\boxed{\text{STEP 2}}$ MAXIMIZE: $|f'''(x)|=\frac{\pi^3}{6^3}\cos(\pi x/6)$ ← decreasing on I

$M=\frac{\pi^3}{6^3}\cos\left(\frac{\pi}{6}\right)=\frac{\pi^3}{6^3}\sqrt{3}/2$

≈ 0.124315848

for a less precise bound you could use 1 here which would still get full credit.

$\boxed{\text{STEP 3}}$ TAYLOR'S INEQ.

$$|f(x)-T_2(x)|\le\frac{M}{3!}|x-1|^3\le\frac{M}{6}(0.1)^3$$

$$\approx\boxed{0.000020719}$$

3 $f(x)=x\ln(x)\quad b=1 \quad\rightarrow\quad f(1)=1\ln(1)=0$

$f'(x)=\ln(x)+x\frac{1}{x}=\ln(x)+1 \quad\rightarrow\quad f'(1)=\ln(1)+1=1$

$f''(x)=\frac{1}{x} \quad\rightarrow\quad f''(1)=1/1=1$

(a) $$\boxed{\begin{aligned}T_2(x)&=0+1(x-1)+\frac{1}{2!}1(x-1)^2\\ T_2(x)&=(x-1)+\frac{1}{2}(x-1)^2\end{aligned}}$$

(b) $0.9\ln(0.9)\approx(0.9-1)+\frac{1}{2}(0.9-1)^2$

$=-0.1+\frac{1}{2}0.01=\boxed{-0.095}$

(c) Interval : $I = [0.9, 1]$

STEP 1 Next Derivative : $f'''(x) = -\frac{1}{x^2}$

STEP 2 Maximize : $|f'''(x)| = \frac{1}{x^2}$ ← decreasing on I

$M = \frac{1}{0.9^2} = \frac{1}{0.81} \approx 1.234567901$

STEP 3 Taylor's Inequality

$|f(x) - T_2(x)| \leq \frac{M}{3!}|x-1|^3 \leq \frac{M}{6}(0.1)^3 \approx \boxed{0.00020576}$

4 $f(x) = x^3 + x$, $b = 1$ → $f(1) = 2$

$f'(x) = 3x^2 + 1$ $\quad f'(1) = 4$

$f''(x) = 6x$ $\quad f''(1) = 6$

(a) $T_2(x) = 2 + 4(x-1) + \frac{1}{2!}6(x-1)^2$

$\boxed{T_2(x) = 2 + 4(x-1) + 3(x-1)^2}$

(b) ERROR BOUND

$f'''(x) = 6$ for all x in any interval

so $M = 6$

Taylor's Inequality tells us

$|T_2(x) - f(x)| \leq \frac{M}{3!}|x-1|^3 = \frac{6}{3!}|x-1|^3 = |x-1|^3$

We want to know which value of x will give an error bound of less than 0.001

$|x-1|^3 < 0.001$, taking the cube root gives

$\Rightarrow |x-1| < 0.1$, which means

$-0.1 < x-1 < 0.1$, and adding 1 gives

$0.9 < x < 1.1$

The interval $\boxed{J = (0.9, 1.1)}$ gives an error bound less than 0.001

5 $\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x^2)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2}$ for all x

Thus,

$$\int_0^2 \sin(x^2)\,dx = \int_0^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{4k+2}\,dx$$

ACTUAL VALUE 0.8047764893

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{4k+3} x^{4k+3}\Big|_0^2$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{(4k+3)} 2^{4k+3}$$

$$\approx \frac{1}{1!}\frac{1}{3}2^3 - \frac{1}{3!}\frac{1}{7}2^7 + \frac{1}{5!}\frac{1}{11}2^{11} - \frac{1}{7!}\frac{1}{15}2^{15}$$

$$= \boxed{0.737123617}$$

6

$$\frac{1}{1+5x} = \sum_{n=0}^{\infty} (-5x)^n = \sum_{n=1}^{\infty} (-1)^n 5^n x^n \quad \text{for} \quad -1 < -5x < 1$$

$$\tfrac{1}{5} > x > -\tfrac{1}{5}$$

$$\frac{1}{3+x} = \frac{1}{3} \frac{1}{1+x/3} = \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n \quad \text{for} \quad -1 < -x/3 < 1$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^n \qquad 3 > x > -3$$

TOGETHER:

$$\frac{1}{1+5x} + \frac{1}{3+x} = \sum_{n=0}^{\infty} \left[(-1)^n 5^n + \frac{(-1)^n}{3^{n+1}}\right] x^n \quad \text{for} \quad -\tfrac{1}{5} < x < \tfrac{1}{5}$$

FIRST FOUR TERMS:

$$\boxed{\left(1 + \tfrac{1}{3}\right) - \left(5 + \tfrac{1}{9}\right)x + \left(5^2 + \tfrac{1}{27}\right)x^2 - \left(5^3 + \tfrac{1}{81}\right)x^3}$$

7

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} \quad \text{for all } x.$$

so

$$x^3 e^{x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n+3} = x^3 + \frac{1}{1!}x^5 + \frac{1}{2!}x^7 + \frac{1}{3!}x^9 + \frac{1}{4!}x^{11} + \dots$$

$$\tfrac{1}{4!} = \boxed{\tfrac{1}{24} = \text{the coefficient of } x^{11}}$$

8

$$\cos(5x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(5x^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 5^{2n} x^{4n} \quad \text{for all } x$$

so

$$x^3 \cos(5x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 5^{2n} x^{4n+3} \quad \text{for all } x$$

$$\int x^3 \cos(5x^2) = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 5^{2n} x^{4n+3}\, dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 5^{2n} \frac{1}{(4n+4)} x^{4n+4}}$$

9 $f(x) = \ln(3+2x^2)$

(a)

$$f'(x) = \frac{4x}{3+2x^2} = \frac{4x}{3} \frac{1}{1+\frac{2}{3}x^2}$$

$$\frac{1}{1+\frac{2}{3}x^2} = \sum_{n=0}^{\infty}\left(-\frac{2}{3}x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^n} x^{2n} \quad \text{for} \quad -1 < \tfrac{2}{3}x^2 < 1$$

$$-\tfrac{3}{2} < x^2 < \tfrac{3}{2}$$

$$-\sqrt{\tfrac{3}{2}} < x < \sqrt{\tfrac{3}{2}}$$

so

$$\boxed{\frac{4x}{3(1+\frac{2}{3}x^2)} = \sum_{n=0}^{\infty} \frac{4(-1)^n 2^n}{3^{n+1}} x^{2n+1}}$$

(b) $f(x) = \int \frac{4x}{3(1+\frac{2}{3}x^2)}dx = C + \sum_{n=0}^{\infty} \frac{4(-1)^n 2^n}{3^{n+1}(2n+2)} x^{2n+2} = \ln(3+2x^2)$

$x=0 \Rightarrow f(0) = \ln(3) \Rightarrow C = \ln(3)$

Thus,

$$\boxed{\ln(3+2x^2) = \ln(3) + \sum_{n=0}^{\infty} \frac{4(-1)^n 2^n}{3^{n+1}(2n+2)} x^{2n+2}}$$

(c) Interval: $\left(-\sqrt{3/2}, \sqrt{3/2}\right)$ $\boxed{\text{Radius} = \sqrt{3/2}}$

(d) $T_4(x) = \ln(3) + \frac{4}{6}x^2 - \frac{\cancel{4}\cdot 2}{9\cdot\cancel{4}}x^4 = \ln(3) + \frac{2}{3}x^2 - \frac{2}{9}x^4$

Interval: $(-a, a)$

$f'(x) = \frac{4x}{3+2x^2}$

$f''(x) = \frac{(3+2x^2)4 - 4x\cdot 4x}{(3+2x^2)^2} = \frac{12-8x^2}{(3+2x^2)^2}$

$f'''(x) = \frac{(3+2x^2)^2(-16x) - (12-8x^2)2(3+2x^2)4x}{(3+2x^2)^4}$

TOO MESSY TO BE AN EXAM PROBLEM,
THIS WAS INCLUDED IN THE WORKSHEET BY MY ERROR
(THIS QUESTION ACTUALLY WENT WITH A DIFFERENT FUNCTION)
TO DO THIS YOU WOULD NEED TO COMPUTE $f^{(5)}(x)$.

$\boxed{10}$ $\vec{v} = \langle a, b, c\rangle$

Want (i) $\vec{v}$ is orthogonal to $\langle 2,1,4\rangle \Rightarrow \langle a,b,c\rangle \cdot \langle 2,1,4\rangle = 0$

So $\boxed{2a+b+4c=0}$

(ii) $\vec{v} \times \langle 1,2,0\rangle = \langle 2,-1,0\rangle \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ 1 & 2 & 0 \end{vmatrix} = \langle 2,-1,0\rangle$

$\langle \overset{\mathrm{I}}{b\cdot 0 - 2\cdot c}, \overset{\mathrm{II}}{1\cdot c - a\cdot 0}, \overset{\mathrm{III}}{2\cdot a - 1\cdot b}\rangle = \langle 2,-1,0\rangle$

So $\boxed{\mathrm{I}}$ $-2c = 2 \Rightarrow \boxed{c=-1}$

$\boxed{\mathrm{II}}$ $\boxed{c=-1}$

$\boxed{\mathrm{III}}$ $\boxed{2a-b=0} \Rightarrow 2a = b$

(i) & (ii) give $2a+b+4c=0$

$2a+2a-4=0 \Rightarrow 4a=4$ $\boxed{a=1}$

$b=2a \Rightarrow \boxed{b=2}$

$$\boxed{\vec{v} = \langle 1,2,-1\rangle}$$

11 (a) Equating components: (i) $2t = 2-2u \Rightarrow t = 1-u$

(ii) $0 = 3u \Rightarrow \boxed{u=0}$ so $\boxed{t=1}$

(iii) $4-4t=0$ ✓

$t=1 \Rightarrow \boxed{(x,y,z) = (2,0,0)}$ ✓

$u=0 \Rightarrow (x,y,z) = (2,0,0)$

(b) Find two vectors parallel to the plane

(or find 3 pts: $P(0,0,4)$, $Q(2,0,0)$, $R(0,3,0)$)

Vectors: $\vec{v}_1 = \langle 2,0,-4\rangle$, $\vec{v}_2 = \langle -2,3,0\rangle$

$\vec{n} = \langle 2,0,-4\rangle \times \langle -2,3,0\rangle = \langle 0--12, 8-0, 6-0\rangle$

$\vec{n} = \langle 12,8,6\rangle$ ✓

$\langle 12,8,6\rangle \cdot \langle x-2, y, z\rangle = 0$

$12(x-2) + 8y + 6z = 0 \Rightarrow \boxed{12x + 8y + 6z = 24}$

12 Find two points of intersection

(or find one point and cross the normals to get direction)

$z=0 \Rightarrow \begin{matrix} x+y=1 \\ 3x-y=1 \end{matrix} \Rightarrow 4x = 2 \Rightarrow x = 1/2 \quad y = 1/2$

$(1/2, 1/2, 0)$

$y=0 \Rightarrow \begin{matrix} x+2z=1 \\ 3x+4z=1 \end{matrix} \Rightarrow \begin{matrix} 2x+4z=2 \\ 3x+4z=1 \end{matrix} \Rightarrow -x = 1 \quad x=-1 \Rightarrow z=1$

$(-1,0,1)$

DIRECTION VECTOR: $\vec{v} = \langle 1/2 - -1, 1/2 - 0, 0-1\rangle = \langle 3/2, 1/2, -1\rangle$

(or any ~~nonzero~~ multiple of this vector)

say $\vec{v} = \langle 3,1,-2\rangle$ for simplicity.

$\vec{r}_0 = \langle -1,0,1\rangle$ (or any point of intersection)

$\langle x,y,z\rangle = \langle -1,0,1\rangle + t\langle 3,1,-2\rangle$

$\boxed{x = -1+3t, \ y = t, \ z = 1-2t}$

13 Through origin $\vec{r}_0 = \langle 0,0,0\rangle$

$\vec{n} = \langle a,b,c\rangle$ is perpendicular to both $\langle 5,-1,1\rangle$ and $\langle 3,2,-3\rangle$

$\langle 5,-1,1\rangle \times \langle 3,2,-3\rangle = \langle 3-2, 3--15, 10--3\rangle = \langle 1,17,13\rangle$

$\langle 1,17,13\rangle \cdot \langle x-0, y-0, z-0\rangle = 0$

$\boxed{x + 17y + 13z = 0}$