Worksheet 5 Math 126
(best if done in small groups)

Precalculus courses do not always include material on hyperbolas. This worksheet covers material that will be useful in section 12.6, for example.

A curve given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

is called a **hyperbola**. Actually there are two curves which satisfy equation (1), and they are called the "branches" of the hyperbola. We may assume that both a and b are positive.

1. (a) Why is $|x| \geq |a|$ on this hyperbola? What does this statement mean about the graph of the hyperbola?
   (b) Show that the branches of the hyperbola lie between the two lines $y = bx/a$ and $y = -bx/a$ and are asymptotic to both lines.
   Hint: First assume $y \geq 0$ and $x \geq a > 0$. Solve equation (1) for y and compare with $y = bx/a$. Use symmetry to find the graph in the other three quadrants.
   (c) Graph the hyperbola $4x^2 - 9y^2 = 1$. Also draw the asymptotic lines, and label the closest points to $(0,0)$ (these are called "vertices").
   (d) A curve given by the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{2}$$

   is also a hyperbola. How is the graph of (2) related to the graph of (1)? You can tell which graph is which by noting that y cannot equal zero in (2) and x cannot equal zero in (1).

A space vehicle passing by a planet has a path which is (approximately) a hyperbola. The planet actually can act like a sling-shot for the space vehicle, increasing the speed of the space vehicle as it passes by.

In the next two problems we will investigate quadratic surfaces that are related to hyperbolas. One way to visualize a quadratic surface is to find traces, or cross sections. This means treating one variable as a constant, and then finding the shape of the curve that satisfies the equation.

2. For example:

$$z = 4x^2 - 9y^2$$

   is called a **hyperbolic paraboloid**. The trace with z height equal to 1 is the hyperbola from problem 1c. So if you lift the hyperbola up to z height 1, then it will lie on the surface. To find the trace at other z heights, fix a value of z then draw the curve

given by the equation. For each value of z (except 0), you will get a hyperbola, but a different hyperbola for each value of z.

(a) What happens at height $z = 0$?

Additional information about the surface can be found by fixing the variable y to get a trace perpendicular to the y-axis.

(b) What curve do you get when $y = 1$?
(c) What happens for other values of y?
(d) What happens when you fix $x = 2$ on the surface?
(e) Can you see where the surface gets its name?
(f) Try to sketch a picture of the surface. It should look like a mountain pass or saddle.

3. **Hyperboloid** of one sheet:
$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1.$$

What are the horizontal traces? (z constant)
What are the vertical traces perpendicular to the x-axis?
What are the vertical traces perpendicular to the y-axis?
Describe the surface. Try to picture how the horizontal traces change when z changes by using one or more vertical traces.

4. (Optional) Here is the geometric description for the hyperbola. Let $c^2 = a^2 + b^2$. The hyperbola is the set of points (x, y) such that the distance from $(c, 0)$ to (x, y) minus the distance from $(-c, 0)$ to (x, y) equals $\pm 2a$. Use the distance formula to derive equation (2) from this geometric description. Hints: Square the difference of the distances, move everything except the remaining square root to one side, then square again. A very similar derivation of the equation for an ellipse appears on p. 686 in the Stewart text