Taylor Polynomials Overview

We found that we can approximate functions $f(x)$ with polynomials based at $x = b$ in the following way.

$$T_1(x) = \sum_{k=0}^{1} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b).$$

$$T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2.$$

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + \cdots + \frac{f'''(b)}{3!}(x-b)^3.$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + \cdots + \frac{f^{(n)}(b)}{n!}(x-b)^n.$$

Taylor inequalities

And we found that we can get a bound on the error in the following way.

$$\text{ERROR} = |f(x) - T_1(x)| \leq \frac{M}{2!}|x-b|^2 \qquad \text{, where } |f''(x)| \leq M.$$

$$\text{ERROR} = |f(x) - T_2(x)| \leq \frac{M}{3!}|x-b|^3 \qquad \text{, where } |f'''(x)| \leq M.$$

$$\text{ERROR} = |f(x) - T_3(x)| \leq \frac{M}{4!}|x-b|^4 \qquad \text{, where } |f^{(4)}(x)| \leq M.$$

$$\text{ERROR} = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1} \quad \text{, where } |f^{(n+1)}(x)| \leq M.$$

We asked the following error questions:

1. Given a fixed n and a fixed interval, find the error bound.
2. Given a fixed n and an error, find an interval with an error bound less than the given error.
3. Given a fixed interval and an error, find a number n with an error bound less than the given error.

Taylor Series Overview

Then we started looking for patterns in the Taylor series for some of our standard functions. We found:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \quad \text{, for all } x.$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1} = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots \quad \text{, for all } x.$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k} = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \cdots \quad \text{, for all } x.$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad \text{, for } -1 < x < 1.$$

We learned:

1. We can substitute in for x in any of these (and in the last case, find the new interval of convergence).
2. We can integrate and differentiate and get a new Taylor series with the same interval of convergence.

Some notable examples include (each of the series below have an interval of convergence of $-1 < x < 1$):

$$-\ln(1-x) = \int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^{\infty} \frac{1}{k+1}x^{k+1} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots.$$

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}x^{2k+1} = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \cdots.$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} kx^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

$$\frac{2}{(1-x)^3} = \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \sum_{k=0}^{\infty} k(k-1)x^{k-2} = 2 + 2\cdot 3x + 3\cdot 4x^2 + \cdots.$$