Math 126 C - Spring 2009
Mid-Term Exam Number One
April 21, 2009
Solutions

1. (a) *Find the equation of the plane P containing the point (1,2,3) which is parallel to the plane containing the points (0,3,4), (3,2,1), and (5,4,2).*

   We first find two vectors extending between two pairs of points given.

   Two such vectors are $\langle 3, -1, -3\rangle$ and $\langle 5, 1, -2\rangle$.

   Taking their cross product we have the vector $\langle 5, -9, 8\rangle$.

   This is the normal vector to the plane which is parallel to the plane P, and hence is the normal vector for the plane P.

   Since P contains the point $(1, 2, 3)$, an equation for P is

   $$5(x - 1) - 9(y - 2) + 8(z - 3) = 0$$

   which can be, optionally, simplified to

   $$5x - 9y + 8x = 11.$$

   (b) *Give an example of a line contained in plane P.*

   There are very many reasonable approaches to this problem. One is to take the vector $\langle 3, -1, 3\rangle$, known to be parallel to P, as the direction vector for the line. Then, taking the point $(1, 2, 3)$, known to be in P, we have the line

   $$x = 1 + 3t, y = 2 - t, z = 3 + 3t$$

2. *Thoroughly describe the surface defined as the set of points which are twice as far from the z-axis as they are from the xy-plane.*

   The distance from the point (x, y, z) to the xy-plane is $|z|$.

   The distance from the point (x, y, z) to the z-axis is $\sqrt{x^2 + y^2}$.

   Thus, the surface defined is the set of points satisfying the equation

   $$\sqrt{x^2 + y^2} = 2|z|.$$

   Squaring this equation yields

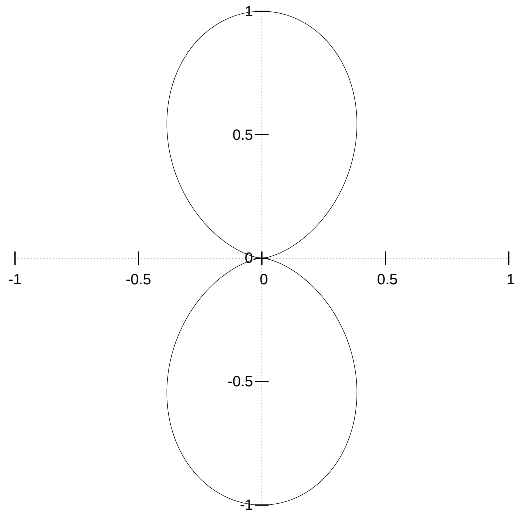
   $$x^2 + y^2 = 4z^2$$

   or

   $$\frac{x^2}{4} + \frac{y^2}{4} - z^2 = 0.$$

   This we recognize as the equation of a cone. This is a cone with apex at the origin and axis the z-axis. Traces parallel to the xy-plane are circles, while traces parallel to the xz-plane or the yz-plane are hyperbolas, except for those passing through the origin (such traces are pairs of lines through the origin).

3. *The curve defined by the polar equation $r = \sin^2\theta$ is shown in the figure below.*



(a) *Find the slope of the tangent line to the curve at the point where $\theta = \dfrac{\pi}{4}$.*

We have $x = r\cos\theta = \cos\theta\sin^2\theta$ and $y = r\sin\theta = \sin^3\theta$. From this we are able to conclude, after simplification, that

$$\frac{dy}{dx} = \frac{3\sin\theta\cos\theta}{2\cos^2\theta - \sin^2\theta}.$$

Taking $\theta = \frac{\pi}{4}$, we find

$$\frac{dy}{dx} = 3.$$

(b) *What is the maximum x-coordinate for a point on this curve?* We see, assisted by the figure, that where the maximal x-coordinate occurs,

$$\frac{dx}{d\theta} = 0.$$

Thus we need to solve

$$\sin\theta(2\cos^2\theta - \sin^2\theta) = 0.$$

Since $\sin\theta = 0$ results in $r = 0$ and $x = 0$, we need only concern ourselves with the other factor. Since

$$2\cos^2\theta - \sin^2\theta = 2\cos^2\theta - (1 - \cos^2\theta) = 3\cos^2\theta - 1$$

we may conclude that

$$\cos\theta = \frac{\pm 1}{\sqrt{3}}.$$

Since $x = \cos\theta(1-\cos^2\theta)$, we conclude that the maximum x-coordinate is

$$\frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right) = \frac{2}{3\sqrt{3}} \approx 0.384900179....$$

4. *Where does the line which passes through the points $(0,5,-3)$ and $(1,2,8)$ intersect the plane $x-3y+4z=11$?*

   We may begin by finding parametric equations for the line. This will do:

$$x = t, y = 5-3t, z = -3+11t.$$

   Then, we seek a solution to

$$t - 3(5-3t) + 4(-3+11t) = 11.$$

   The solution is $t = \frac{19}{27}$. Hence the point is $\left(\frac{19}{27}, \frac{26}{9}, \frac{128}{27}\right)$.

5. *Consider the curve with the vector equation*

$$\vec{r}(t) = \langle t^2, 2t^2 - t, 3t - t^2 \rangle$$

   Is there a point on this curve where the tangent line is parallel to the vector $\langle 20, 38, -14\rangle$? If so, find the point. If not, explain why.

   The tangent vector is

$$\vec{r}(t) = \langle 2t, 4t-1, 3-2t \rangle.$$

   If this vector is parallel to $\langle 20, 38, -14\rangle$ for some t, then there exists a scalar k such that

$$2t = 20k \text{ and } 4t - 1 = 38k.$$

   Solving this pair of equations simultaneously yields $k = 1/2$ and $t = 5$.

   Checking the z-components, we find that $3 - 2(5) = -7 = \frac{1}{2}(-14)$, so the direction vector at $t = 5$ is, indeed, $1/2$ times the vector $\langle 20, 38, -14\rangle$, and so the direction vector is parallel to $\langle 20, 38, -14\rangle$.

   The sought point on the line is thus $\langle 25, 45, -10\rangle$.