1. (10 points) A particle's position in two dimensions is given by the parametric equations

$$x = \ln(t)$$
$$y = 2t^3 - 21t^2 + 60t,$$

where $t > 0$.

(a) (5 points) Find all points (x, y) where the parametric curve has a horizontal tangent.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2 - 42t + 60}{1/t}$$

$$dy/dt = 0 \Rightarrow \left.\begin{array}{l} 6t^2 - 42t + 60 = 0 \\ t^2 - 7t + 10 = 0 \\ (t-2)(t-5) = 0 \end{array}\right\} \underline{t = 2} \text{ OR } \underline{t = 5}$$

POINTS $\boxed{\begin{array}{l}(x,y) = (\ln(2), 52) \\ (x,y) = (\ln(5), 25)\end{array}}$

(b) (5 points) Find the velocity and acceleration vectors at the point $(0, 41)$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle \tfrac{1}{t}, 6t^2 - 42t + 60 \rangle$$

$$\vec{a}(t) = \vec{r}\,''(t) = \langle -\tfrac{1}{t^2}, 12t - 42 \rangle$$

$$\left.\begin{array}{l} x = 0 \\ y = 41 \end{array}\right\} \Rightarrow \begin{array}{l} 0 = \ln(t) \quad \underline{t = 1} \\ 41 = 2 - 21 + 60 \checkmark \end{array}$$

$$\boxed{\begin{array}{l}\vec{v}(1) = \langle 1, 24 \rangle \\ \vec{a}(1) = \langle -1, -30 \rangle\end{array}}$$

2. (8 points) Let r be an unknown constant.

P Q R

(a) (5 points) Find the equation of the plane through the points $(0, 2, 0)$, $(1, 0, 0)$, and $(0, 1, r)$. Write your answer in the form $ax + by + cz + d = 0$.
(Your answer will be the equation of a plane and will involve the constant r.)

$\vec{PQ} = \langle 1, -2, 0 \rangle \qquad \vec{PR} = \langle 0, -1, r \rangle$

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & r \\ 1 & -2 & 0 \end{array}$$

$\vec{PR} \times \vec{PQ} = \langle 2r, r, 1 \rangle = \vec{n}$

$\vec{r}_0 = \langle 1, 0, 0 \rangle$

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$\langle 2r, r, 1 \rangle \cdot \langle x-1, y, z \rangle = 0$

$$\boxed{\begin{array}{c} 2r(x-1) + ry + z = 0 \\ 2rx + ry + z - 2r = 0 \end{array}}$$

(b) (3 points) If $(3, 2, 1)$ is also a point on the plane, what is the value of r?

$\left.\begin{array}{l} x = 3 \\ y = 2 \\ z = 1 \end{array}\right\}$ satisfies the plane equation

$\Rightarrow \quad 2r(3) + r(2) + (1) - 2r = 0$

$\Rightarrow \quad 6r + 2r + 1 - 2r = 0$

$\Rightarrow \quad 6r + 1 = 0$

$6r = -1$

$$\boxed{r = -\tfrac{1}{6}}$$

3. (12 points) Let $\mathbf{r}(t) = \langle 3\cos(t), 5\sin(t), 4\cos(t)\rangle$.

(a) (4 points) Show that the unit tangent vector $\mathbf{T}(t)$ is orthogonal to $\mathbf{r}''(t)$ for all t.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \qquad \vec{T}(t) \cdot \vec{r}''(t) = 0 \iff \vec{r}'(t) \cdot \vec{r}''(t) = 0$$

$$\vec{r}'(t) = \langle -3\sin(t), 5\cos(t), -4\sin(t)\rangle$$

$$\vec{r}''(t) = \langle -3\cos(t), -5\sin(t), -4\cos(t)\rangle$$

$$\vec{r}'(t) \cdot \vec{r}''(t) = 9\sin(t)\cos(t) - 25\cos(t)\sin(t) + 16\sin(t)\cos(t)$$

$$= 0 \quad \checkmark$$

(b) (4 points) Find the parametric equations for the tangent line to the curve at $t = \frac{\pi}{2}$.

$$\vec{r} = \vec{r}_0 + t\vec{v} \qquad \vec{v} = \vec{r}'(\tfrac{\pi}{2}) = \langle -3, 0, -4\rangle$$

$$\vec{r}_0 = \vec{r}(\pi/2) = \langle 0, 5, 0\rangle$$

$$\langle x, y, z\rangle = \langle 0, 5, 0\rangle + t\langle -3, 0, -4\rangle$$

$$\boxed{\begin{aligned} x &= -3t \\ y &= 5 \\ z &= -4t \end{aligned}}$$

(c) (4 points) Find the curvature $\kappa(t)$.

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -3\sin(t), 5\cos(t), -4\sin(t)\rangle}{\sqrt{9\sin^2(t) + 25\cos^2(t) + 16\sin^2(t)}}$$

$$= \langle -\tfrac{3}{5}\sin(t), \cos(t), -\tfrac{4}{5}\sin(t)\rangle$$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\sqrt{\frac{9}{25}\sin^2(t) + \cos^2(t) + \frac{16}{25}\sin^2(t)}}{5} = \boxed{\frac{1}{5}}$$

4. (10 points) The acceleration for a particle is given by the vector function $\mathbf{a}(t) = \langle t, 0, -1\rangle$ for $t \geq 0$. The initial position of the particle $\mathbf{r}(0) = \langle 1, 0, 1\rangle$ and the initial velocity is $\mathbf{v}(0) = \langle -2, 1, 0\rangle$.

(a) (4 points) Find the position vector $\mathbf{r}(t)$.

$$\vec{v}(t) = \langle \tfrac{1}{2}t^2 + c_1,\ c_2,\ -t + c_3\rangle$$

$$\vec{v}(0) = \langle -2, 1, 0\rangle \Rightarrow c_1 = -2,\ c_2 = 1,\ c_3 = 0$$

$$\vec{v}(t) = \langle \tfrac{1}{2}t^2 - 2,\ 1,\ -t\rangle$$

$$\vec{r}(t) = \langle \tfrac{1}{6}t^3 - 2t + d_1,\ t + d_2,\ -\tfrac{1}{2}t^2 + d_3\rangle$$

$$\vec{r}(0) = \langle 1, 0, 1\rangle \Rightarrow d_1 = 1,\ d_2 = 0 \quad d_3 = 1$$

$$\boxed{\vec{r}(t) = \langle \tfrac{1}{6}t^3 - 2t + 1,\ t,\ -\tfrac{1}{2}t^2 + 1\rangle}$$

(b) (2 points) Will the particle ever be located at the origin? If not, explain why. If so, give all times when the particle is at the origin.

$$\left.\begin{aligned} 0 &= \tfrac{1}{6}t^3 - 2t + 1 \\ 0 &= t \\ 0 &= -\tfrac{1}{2}t^2 + 1 \end{aligned}\right\}$$

can't happen

$t = 0 \Rightarrow -\tfrac{1}{2}t^2 + 1 = 1 \neq 0$

$\boxed{\text{NEVER AT ORIGIN}}$

(c) (4 points) Find all times when the the tangential component of the acceleration vector is zero.

$$a_T = (\text{speed})' = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|} = \frac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t) \cdot \vec{a}(t)}{\sqrt{(\tfrac{1}{2}t^2 - 2)^2 + 1 + t^2}}$$

So $a_T = 0 \Leftrightarrow \vec{v}(t) \cdot \vec{a}(t) = 0$

$$\Leftrightarrow \langle \tfrac{1}{2}t^2 - 2, 1, -t\rangle \cdot \langle t, 0, -1\rangle = 0$$

$$\Leftrightarrow \tfrac{1}{2}t^3 - 2t + t = 0$$

$$\Leftrightarrow \tfrac{1}{2}t^3 - t = 0$$

$$t(\tfrac{1}{2}t^2 - 1) = 0$$

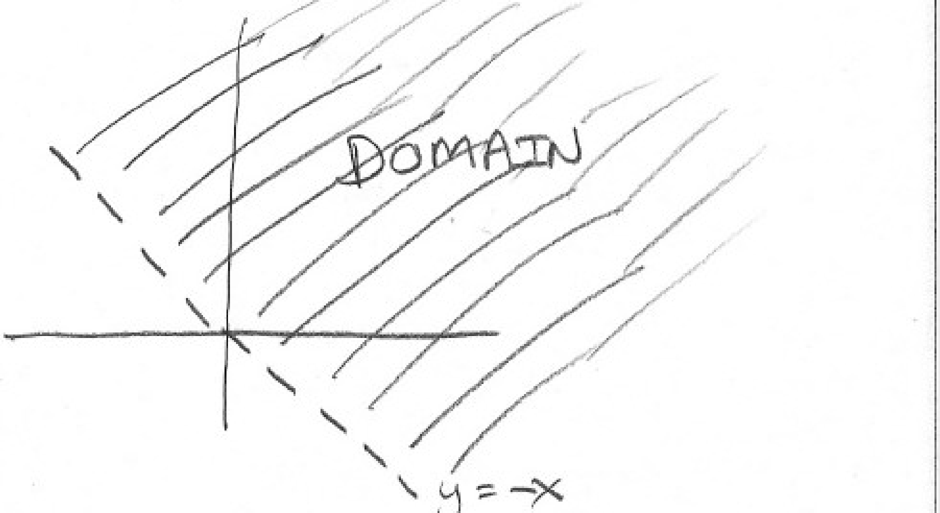
$$\boxed{\begin{aligned} t &= 0 \\ t &= -\sqrt{2} \\ t &= \sqrt{2} \end{aligned}}$$

5. (10 points) Let $f(x, y) = x^2y + x\ln(x + y)$

(a) (2 points) Find and sketch the domain of the function.
Clearly indicate everything that is included in the domain in your sketch.

$x + y > 0$

$\boxed{y > -x}$



(b) (4 points) Compute the partial derivatives of $f_x(x, y)$ and $f_y(x, y)$.

$$\boxed{f_x(x,y) = 2xy + \ln(x+y) + \frac{x}{x+y}}$$

$$\boxed{f_y(x,y) = x^2 + \frac{x}{x+y}}$$

(c) (4 points) Compute the second partial derivatives $f_{xy}(x, y)$ and $f_{yy}(x, y)$

$$\boxed{f_{xy}(x,y) = 2x + \frac{1}{x+y} - \frac{x}{(x+y)^2}}$$

$$\boxed{f_{yy}(x,y) = -\frac{x}{(x+y)^2}}$$