1. (12 points) Let $\vec{a} = <3, -1, 2>$ and $\vec{b} = 5\vec{i} - 7\vec{j} + 2\vec{k}$.

(a) (4 points) The equation for a sphere is given by $x^2 + (y-3)^2 + z^2 = 12 + 12x - 4z$. Find the distance from the origin to the center of the sphere.

$$x^2 - 12x + 36 + (y-3)^2 + z^2 + 4z + 4 = 12 + 36 + 4$$

$$(x-6)^2 + (y-3)^2 + (z+2)^2 = 52$$

center $= (6, 3, -2)$

$$\text{distance} = \sqrt{6^2 + 3^2 + (-2)^2} = \sqrt{36+9+4} = \sqrt{49} = \boxed{7}$$

(b) (4 points) Find $\vec{a} \cdot \vec{b}$.

$$\vec{a} \cdot \vec{b} = (3)(5) + (-1)(-7) + (2)(2)$$

$$= 15 + 7 + 4 = \boxed{26}$$

(c) (4 points) Find the angle, θ, between the vectors $\vec{a}$ and $\vec{b}$. Give your answer in radians such that $0 \le \theta \le \pi$. (Round your answer to 3 digits after the decimal point.)

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$

$$26 = \sqrt{3^2 + (-1)^2 + 2^2}\sqrt{5^2 + (-7)^2 + 2^2} \cos(\theta)$$

$$26 = \sqrt{14}\sqrt{78} \cos(\theta)$$

$$\boxed{\theta = \cos^{-1}\left(\frac{26}{\sqrt{14}\sqrt{78}}\right) \approx 0.665 \text{ radians}}$$

2. (10 points)

(a) (5 points) Find all values of x so that $\vec{a} =< 1, x, -4 >$ and $\vec{b} =< x, 3, 5 >$ are orthogonal.

$$\vec{a} \cdot \vec{b} = 0 \Rightarrow \langle 1, x, -4 \rangle \cdot \langle x, 3, 5 \rangle = 0$$

$$\Rightarrow x + 3x - 20 = 0$$

$$\Rightarrow 4x = 20$$

$$\Rightarrow \boxed{x = 5}$$

(b) (5 points) Find a **unit** vector that is orthogonal to both $\vec{a} =< 1, 4, 5 >$ and $\vec{b} =< -1, 3, 0 >$.

$$\begin{array}{cccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 5 & 1 & 4 & 5 \\ -1 & 3 & 0 & -1 & 3 & 0 \end{array}$$

$$\vec{a} \times \vec{b} = (0-15)\vec{i} + (-5-0)\vec{j} + (3- -4)\vec{k}$$

$$= -15\vec{i} - 5\vec{j} + 7\vec{k}$$

$$= \langle -15, -5, 7 \rangle = \vec{v}$$

$$|\vec{v}| = \sqrt{15^2 + 5^2 + 7^2} = \sqrt{299}$$

$$\text{unit vector} = \frac{1}{|\vec{v}|}\vec{v}$$

$$= \frac{1}{\sqrt{299}} \langle -15, -5, 7 \rangle$$

$$= \left\langle \frac{-15}{\sqrt{299}}, \frac{-5}{\sqrt{299}}, \frac{7}{\sqrt{299}} \right\rangle$$

ALSO CORRECT $\Rightarrow$ $\left\langle \frac{15}{\sqrt{299}}, \frac{5}{\sqrt{299}}, \frac{-7}{\sqrt{299}} \right\rangle$

3. (12 points) Give the Taylor series for $g(x) = \int_0^x \cos(t^3)\, dt$ based at $b = 0$.

Write your answer using sigma notation, write out the first 3 nonzero terms, and give the open interval of convergence.

$$\cos(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} u^{2k} \quad \text{for } I = (-\infty, \infty)$$

So $$\cos(t^3) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (t^3)^{2k} \quad \text{for } I = (-\infty, \infty)$$

$$\Rightarrow \int_0^x \cos(t^3)\,dt = \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{6t}\,dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int_0^x t^{6k}\,dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{1}{6k+1} t^{6k+1} \Big|_0^x$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{1}{6k+1} x^{6k+1}} \qquad \boxed{I = (-\infty, \infty)}$$

$$= x - \frac{1}{2}\frac{1}{7}x^7 + \frac{1}{24}\frac{1}{13}x^{13} - \cdots$$

$$\boxed{= x - \frac{1}{14}x^7 + \frac{1}{312}x^{13} -}$$

4. (12 points) Give the Taylor series for $f(x) = \dfrac{4x}{5x+1} - xe^{3x}$ based at $b = 0$.

Write your answer using sigma notation, write out the first 3 nonzero terms, and give the open interval of convergence.

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k \qquad I = (-1, 1)$$

$$e^u = \sum_{k=0}^{\infty} \frac{1}{k!} u^k \qquad I = (-\infty, \infty)$$

So $$\frac{4x}{5x+1} = 4x\left(\frac{1}{1-(-5x)}\right) = 4x \sum_{k=0}^{\infty} (-5x)^k \qquad -1 < -5x < 1$$

$$= 4 \sum_{k=0}^{\infty} (-5)^k x^{k+1} \qquad -\frac{1}{5} < x < \frac{1}{5}$$

$$xe^{3x} = x \sum_{k=0}^{\infty} \frac{1}{k!}(3x)^k = \sum_{k=0}^{\infty} \frac{1}{k!} 3^k x^{k+1} \qquad I = (-\infty, \infty)$$

$$\frac{4x}{5x+1} - xe^{3x} = \boxed{4 \sum_{k=0}^{\infty} (-5)^k x^{k+1} - \sum_{k=0}^{\infty} \frac{1}{k!} 3^k x^{k+1} = \sum_{k=0}^{\infty} \left[4(-5)^k - \frac{1}{k!} 3^k\right] x^{k+1}} \qquad \boxed{-\frac{1}{5} < x < \frac{1}{5}}$$

$$= (4-1)x + (4(-5) - 3)x^2 + \left(4(-5)^2 - \frac{1}{2}3^2\right)x^3 + \cdots$$

$$= \boxed{3x - 23x^2 + 95.5x^3 + \cdots}$$

5. (14 points) Let $g(x) = e^{x/2}$ and let $I = [0, 2]$.

(a) (5 points) Find the second Taylor polynomial, $T_2(x)$, for $g(x)$ based at $b = 1$.

$g(x) = e^{x/2}$, $g(1) = e^{1/2}$

$g'(x) = \frac{1}{2}e^{x/2}$, $g'(1) = \frac{1}{2}e^{1/2}$

$g''(x) = \frac{1}{4}e^{x/2}$, $g''(1) = \frac{1}{4}e^{1/2}$

$T_2(x) = g(1) + g'(1)(x-1) + \frac{1}{2}g''(1)(x-1)^2$

$= e^{1/2} + \frac{1}{2}e^{1/2}(x-1) + \frac{1}{2}\cdot\frac{1}{4}e^{1/2}(x-1)^2$

$$\boxed{T_2(x) = e^{1/2} + \frac{1}{2}e^{1/2}(x-1) + \frac{1}{8}e^{1/4}(x-1)^2 = e^{1/2}\left[1 + \frac{1}{2}(x-1) + \frac{1}{8}(x-1)^2\right]}$$

(b) (5 points) Use Taylor's inequality to find a bound on the error for $T_2(x)$ in the interval $I = [0, 2]$.

$|g'''(x)| = \underbrace{\left|\frac{1}{8}e^{x/2}\right|}_{\text{increasing}} \leq \frac{1}{8}e = M$ on $I = [0, 2]$

$M \approx 0.33978$

$|g(x) - T_2(x)| \leq \frac{M}{3!}|x-1|^3 = \frac{e/8}{6}|x-1|^3 \leq \frac{e}{48} = \boxed{0.05663}$

(c) (4 points) Find the first value of n for which the error bound given by Taylor's inequality for $T_n(x)$ in the interval $I = [0, 2]$ is less than 0.001.

$n = 3 \Rightarrow$ $|g^{(4)}(x)| = \left|\frac{1}{16}e^{x/2}\right| \leq \frac{1}{16}e = M$

$\Rightarrow |g(x) - T_3(x)| \leq \frac{M}{4!}|x-1|^4 = \frac{e/16}{24}|x-1|^4$

$\leq \frac{e}{384} \approx \boxed{0.00707}$

$n = 4 \Rightarrow$ $|g^{(5)}(x)| = \left|\frac{1}{32}e^{x/2}\right| \leq \frac{e}{32} = M$

$\Rightarrow |g(x) - T_4(x)| \leq \frac{e/32}{5!}|x-1|^5 = \frac{e/32}{120}|x-1|^5$

$\leq \frac{e}{3840} = \boxed{0.000707886}$

$$\boxed{n = 4}$$