15.1 and 15.2 Review

This review sheet discusses, in a very basic way, the key concepts from these sections. This review is not meant to be all inclusive, but hopefully it reminds you of some of the basics. Please notify me if you find any typos in this review.

1. **15.1 Double Integrals Over Rectangular Regions**: Understand how we define double integrals over rectangular regions and how to approximate them by adding up the volumes of rectangular boxes.

   (a) If R is a rectangular region, then we define

   $$\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

   (b) The integral represents the 'signed' volume 'under' the surface $f(x,y)$. That is, if $f(x,y) \geq 0$ over the region, then $\int\int_R f(x,y)dA$ = 'the volume under the surface and above the xy-plane'.

   If $f(x,y) \leq 0$ over the region, then $\int\int_R f(x,y)dA$ = 'the negative of the volume between the surface and the xy-plane.

   (c) To approximate this value, we do the following:

      i. Break up the region R on the xy-plane into a rectangular m by n grid and we compute the area of one rectangle $= \Delta A$.

      ii. Then we compute the volume of each rectangular box = (height)(area of the base) = $f(x^*, y^*)\Delta A$. To compute the height, select a sample point from the region and plug into your function. There are various ways to pick the sample points:

      - Lower Left Endpoints: After you draw your rectangular grid, use the lower left endpoint of each grid square to plug into your function to compute the height. The same sort of idea goes for Lower Right, Upper Right, and Upper Left endpoints.
      - Midpoint Rule: If you take the midpoint of each rectangular grid as the point you plug in to get the height, then you are using the midpoint rule.

      iii. Finally, sum up all the volumes for the all the rectangular boxes. This gives an approximation for the actual volume of the region.

   The double integral is defined to be the value that this process approaches as you let the rectangles in the rectangular grid become smaller and smaller.

   (d) The average value (that is the average height) of a function $f(x,y)$ over the region R is given by

   $$\frac{1}{\text{the area of R}} \int\int_R f(x,y)dA.$$

   This is analogous to the average value formula from Math 125.

2. **15.2 Iteraged Integrals Over Rectangular Regions**: Know how to evaluate double integrals over rectangular regions by 'integrating twice'.

   (a) Understand the notation:

   $$\int_a^b \int_c^d f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

   $$\int_c^d \int_a^b f(x,y)dydx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$$

In both the cases above, a and b refer to the x variable, and c and d refer to the y variable. In other words, we are integrating over the rectangular region

$$a \leq x \leq b \qquad c \leq y \leq d$$

which can also be stated as the region $R = [a, b] \times [c, d]$. For a rectangular region, you can either integrate with respect to x first or with respect to y first, provided that you integrate x and y over the appropriate bounds. But don't get too much in the habit of switching the order of integration because for non-rectangular regions (which we will cover in 15.3 and 15.4) we have to be more careful when making this switch.

(b) The fact that you can use an interated integral to evaluate a double integral and that it doesn't matter which order of integration is valid due to Fubini's Theorem. In fact, we are only allowed to use the iterated integral under certain conditions on R. Luckily, for us, nearly every natural function that we encounter in this course satisfies these conditions. However, you should be aware that their are very special cases where iterated integrals are not allowed (read Fubini's theorem on page 991).

(c) To use iterated integrals. First evaluate the 'inside' by treated the outside variable as a constant. Then evaluate the 'outside' integral. Some notes:

- After you have finished integrating with respect to the 'inside' variable, that variable should be gone. That is, you should have used the endpoints to evaluate that variable as specific values. And only the outside variable should be present.
- Just like with partial derivatives, if you are integrating with repect to x, then you treat y as a constant (and vice versa).
- Iterated integration is purely computational, we will do some examples in class. There are also several examples in the book. If you are stuck on a problem, it is most likely that you simply don't remember a specific integration techniques from Math 125. In your homework, I'm sure you will need to use u-substitution and integration-by-parts, so you should be looking to use these if you get stuck. You are welcome to look at my Math 125 Review materials for information about integration techniques.