12.1-12.4 Review

This review sheet discusses, in a very basic way, the key concepts from these sections. This review is not meant to be all inclusive, but hopefully it reminds you of some of the basics. Please notify me if you find any typos in this review.

1. **12.1: Three Dimensional Coordinates** - The goal here is to try to get a little practice with three dimensional coordinates.

   - The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

   $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

   - The equation of a sphere with center $C(h, k, l)$ and radius r is

   $$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.$$

   - In the homework, you will be asked to use the distance formula and the equation of the sphere to answer various questions about 3-D points. Here are some facts you may need to use in interpreting these questions:

     (a) An *equilateral triangle* is a triangle with equal distances between all of its points. To verify that something is an equilateral triangle, you must find all three distances and see that they are the same.

     (b) The *pythagorean theorem* states that a right triangle with legs, a and b, and hypotenuse, c, satisfies $a^2 + b^2 = c^2$. In fact, this relationship only holds for right triangles. So if you have the lengths of three sides of a triangle, (a, b, and c, c being the largest), then the triangle is a right triangle if and only if $a^2 + b^2 = c^2$.

     (c) To find if *three points are on the same line*, then you have a few choices:

       i. You can draw the points and see if you can tell from the graph.

       ii. You can read ahead to Section 12.5, where it tells you how to find equations for lines.

       iii. You can find the distances between all three points and try to makes conclusions from this. (In particular, one of the distances will be equal to the sum of the other two if they are on the same line.)

2. **12.2: Vectors** - Understand how to work with vectors.

   - *The Vector Concept*: A *vector* is a quantity that has both magnitude and direction. Typically, we draw a vector as an arrow that starts at the origin (However, as long as the arrow has the same length and direction, it doesn't matter where we draw it.) Here are some standard operations:

     (a) *Representations* - We will typically write a vector in one of two ways, *bracket notation* or *standard basis notation*. For example, $< 2, 3, -5 >$ is the same as $2\mathbf{i} + \mathbf{3j} - \mathbf{5k}$. Here $\mathbf{i} =< \mathbf{1}, \mathbf{0}, \mathbf{0} >$, $\mathbf{j} =< \mathbf{0}, \mathbf{1}, \mathbf{0} >$, and $\mathbf{k} =< \mathbf{0}, \mathbf{0}, \mathbf{1} >$ are called the standard basis vectors.

     (b) *Magnitude and Angle* - The magnitude and the angle of a vector can be determined by follows:

       – The *magnitude* of a vector $\mathbf{v} =< \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} >$ is given by $|\mathbf{v}| = \sqrt{\mathbf{v_1^2} + \mathbf{v_2^2} + \mathbf{v_3^2}}$. Two dimensional magnitude is computed in a similar way.

       – The *angle* that the vector $\mathbf{v} =< \mathbf{v_1}, \mathbf{v_2} >$ makes with the positive x-axis can be determined by drawing a right triangle and using trigonometric functions. For example, in the first quadrant, $\sin(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{v_2}{v_1}$, so $\theta = \sin^{-1}(\frac{v_2}{v_1})$.

(c) A *unit vector* is a vector with magnitude 1. To get a unit vector in the same direction as $\mathbf{v}$, you must divide by the length. That is,

$$\frac{1}{|\mathbf{v}|}\mathbf{v} = \text{'a unit vector in the same direction as } \mathbf{v}.\text{'}$$

(d) *Scalar multiplication* - If c is a scalar and $\mathbf{v}$ is a vector, then $c\mathbf{v}$ means multiply each coordinate of $\mathbf{v}$ by c. For example, if $c = 5$ and $\mathbf{v} =< -\mathbf{1}, \mathbf{4}, \mathbf{7} >= -\mathbf{i} + \mathbf{4j} + \mathbf{7k}$, then $c\mathbf{v} = \mathbf{5v} =< -\mathbf{5}, \mathbf{20}, \mathbf{35} >= -\mathbf{5i} + \mathbf{20j} + \mathbf{35k}$.

(e) *Vector Addition* - If $\mathbf{u}$ and $\mathbf{v}$ are two vectors, the $\mathbf{u} + \mathbf{v}$ is the vector obtaining by adding the corresponding components of each vector. For example, $< 9, -5, 0 > + < -7, 2, 10 >=< 2, -3, 10 >$.
Graphically, the sum $\mathbf{u} + \mathbf{v}$ is the vector that is obtained by drawing from the origin $\mathbf{u}$ and then drawing $\mathbf{v}$ with the tail of $\mathbf{v}$ at the head of $\mathbf{u}$. It can also be viewed as the diagonal through the parallelogram obtained by drawing $\mathbf{u}$ then $\mathbf{v}$, head-to-tail.

(f) *Application to Forces* - Vectors are a useful tool when considering forces on an object. If $\mathbf{F_1}$ and $\mathbf{F_2}$ are vectors representing the forces on an object, then the resultant force is the sum $\mathbf{F_1} + \mathbf{F_2}$. The *tension* in a wire is merely the forces acting on each wire. Please read Example 7 on page 804 of your text for an example of how to find tension.

3. **12.3: The Dot Product** - Be able to find the dot product of two vectors and make sure you know what it gives you.

- The *dot product* of the vectors $\mathbf{a} =< \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3} >$ and $\mathbf{b} =< \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3} >$ is given by

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a_1 b_1} + \mathbf{a_2 b_2} + \mathbf{a_3 b_3},$$

  which is the **scalar** that is given by adding up the products of the components of each vector.

- The dot product gives information about the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. In particular,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta),$$

  where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ with $0 \leq \theta \leq \pi$. Another way to write this is

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|},$$

  provided that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors.

- Perhaps our biggest application of the dot product is *orthogonality.*

$$\mathbf{a} \text{ and } \mathbf{b} \text{ are } \textit{orthogonal}, \text{ or } \textit{perpendicular}, \text{ if and only if } \mathbf{a} \cdot \mathbf{b} = \mathbf{0}.$$

- Sometimes it is useful to project one vector onto another.

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^{\mathbf{2}}}\mathbf{a} = \text{ 'vector projection of } \mathbf{b} \text{ onto } \mathbf{a}.\text{'}$$

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \text{ 'scalar projection of } \mathbf{b} \text{ onto } \mathbf{a}\text{'} = \text{'the length of the projection vector'}.$$

- *Application to Work* - If a constant force $\mathbf{F}$ is applied at an angle of θ along a *displacement vector*, $\mathbf{D}$, the *work* done by the force is defined as

$$W = (|\mathbf{F}| \cos(\theta))|\mathbf{D}| = \mathbf{F} \cdot \mathbf{D}.$$

4. **12.4: Cross Product** - Understand how to compute the cross product and what you can use it.

- The *cross product* of two vectors **a** and **b** yields a new vector that is *orthogonal*, or *perpendicular*, to both **a** and **b**. In other words, it gives a vector that points out, in a perpendicular way, form the triangle determined by **a** and **b**.
- The *cross product* of $\mathbf{a} = < \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3} >$ and $\mathbf{b} = < \mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3} >$ is given by

$$\mathbf{a \ x \ b} = < \mathbf{a_2 b_3} - \mathbf{a_3 b_2}, \mathbf{a_3 b_1} - \mathbf{a_1 b_3}, \mathbf{a_1 b_2} - \mathbf{a_2 b_1} > .$$

  Here are some other ways to remember this:

$$\mathbf{a \ x \ b} = \text{` the determinant of } \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \text{'} .$$

  You can also use the technique discussed in class, where we multiply and subtract the appropriate diagonals.
- If θ is the angle between **a** and **b** and $(0 \leq \theta \leq \pi)$, then

$$|\mathbf{a \ x \ b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta).$$

- Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a \ x \ b} = \mathbf{0}.$$