Taylor Notes 1, 2, and 3 Review

This review sheet discusses, in a very basic way, the key concepts from these sections. This review is not meant to be all inclusive, but hopefully it reminds you of some of the basics. Please notify me if you find any typos in this review.

1. **General Comments** - We are building up a theory of how to approximate functions with so-called Taylor polynomials. Initially, the construction of these polynomials and the errors we find may seem unmotivated. However, Taylor polynomials are an essential tool for exploring functions in science and engineering. They also provide an introduction to infinite series which paves the way for the study of Fourier Series (a vital tool for scientists). So I ask for you to bear with me for the first couple lectures as we build up the necessary machinery to study Taylor series.

2. **Essentials** - After TN 1, 2, and 3, you should definitely know the following big four concepts. I discuss each of these concepts, in order, below.

   - Given $f(x)$ and b, find the Taylor polynomials of $f(x)$ based at b. That is, find $T_1(x)$, $T_2(x)$, *etc.*
   - Given $f(x)$, b and an interval $I = [c, d]$ (usually this interval has b at the midpoint), give a bound for the approximation error in using $T_n(x)$. That is, find a number B so that $|f(x) - T_n(x)| \leq B$ (you have to use Taylor's inequality).
   - Given $f(x)$, b, and an error bound, find an interval so that the approximation error is less than the error bound. That is, find $I = [c, d]$ so that $|f(x) - T_n(x)| \leq$ 'The Given Error Bound'.
   - Given $f(x)$, b, an error bound, and an interval $I = [c, d]$, find an integer, n, so that the approximation error is less than the error bound. That is, find n so that $|f(x) - T_n(x)| \leq$ 'The Given Error Bound in the Interval'.

3. **Finding $T_1(x)$, $T_2(x)$, *etc* And the Corresponding Error Formulas**

   - For each Taylor polynomial, I give the expanded form, then I rewrite it in sigma notation.

$$T_1(x) = \sum_{k=0}^{1} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b).$$

$$T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2.$$

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3.$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \cdots + \frac{f^{(n)}(b)}{n!}(x-b)^n.$$

   - The error formulas (Taylor inequalities) below are given in the same order as the Taylor polynomial that they correspond to above.

$$\text{ERROR} = |f(x) - T_1(x)| \leq \frac{M}{2}|x-b|^2 \quad \text{, where } |f''(x)| \leq M \text{ on the interval.}$$

$$\text{ERROR} = |f(x) - T_2(x)| \leq \frac{M}{3!}|x-b|^3 \quad \text{, where } |f'''(x)| \leq M \text{ on the interval.}$$

$$\text{ERROR} = |f(x) - T_3(x)| \leq \frac{M}{4!}|x-b|^4 \quad \text{, where } |f^{(4)}(x)| \leq M \text{ on the interval.}$$

$$\text{ERROR} = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1} \quad \text{, where } |f^{(n+1)}(x)| \leq M \text{ on the interval.}$$

4. **Using the Taylor Inequalities To Find an Error on a Given an Interval $I = [c, d]$**

   - First you need to compute the next derivative ($f^{(n+1)}(x)$). That is, for $T_1(x)$ you must compute $f''(x)$, or for $T_3(x)$ you must compute $f^{(4)}(x)$.
   - Then you need to find M. To do this you must find the biggest that $|f^{(n+1)}|$ will ever get from $x = c$ to $x = d$. Here are some ways you can do this:
     (a) In general, you could find the absolute maximum and absolute minimum of $f^{(n+1)}(x)$ on the interval and let M be the larger of these two y-coordinates in absolute value.
     (b) If the function is always increasing or always decreasing, then M can be the larger, in absolute value, of the two y-coordinates corresponding to the endpoints. You can often tell if a function is increasing or decreasing by sketching a graph or taking the derivative.
     (c) Finally, you can try to sketch a graph or use trial and error to see if you can figure out the biggest that $|f^{(n+1)}|$ can be.
   - Once you have M, you just plug into the appropriate error formula. If you are asked for the biggest the error can be, then you should also plug in one of the endpoints for x to get the biggest error on the interval.

5. **Using the Taylor Inequalities To Find an Interval $I = [c, d]$ yielding an approximation having smaller than a given Error**

   - Please note the difference between this section and the last. Here you have to find the c and d given a desired error (such as 0.001). There are ways to be slightly more precise, but the following method is sufficient to answer such questions:
     (a) Pick an interval around b. It is immaterial how you pick the interval. For example if $b = 2$, then you could start with the interval $[1.9, 2.1]$ or $[1.75, 2.25]$.
     (b) For the interval you picked. Compute the error as we have already discussed (compute the next derivative, find the M, *etc*). If it is smaller than the given error, then you are done (and you chose wisely in step 1). If it is larger than the given error, then move to the next step.
     (c) Now you can proceed as in the example on page 6 of the Taylor Notes. In particular, write $\frac{M}{(n+1)!}(x-b)^n \leq$ 'the given error' (Use the M from step 2. Then simplify to get $|x - b| \leq$ 'BLAH'. BLAH is the distance required for the interval around b. That is, the interval is $I = [b - BLAH, b + BLAH]$.

6. **Using the Taylor Inequalities To Find an integer, n, yielding an approximation having smaller than a given error within a given interval**

   - Please note the difference between this section and the previous two. Here you are given $I = [c, d]$ and given a desired error (such as 0.001) and you need to find out how far to go out in the Taylor series to get this error. Here I give two methods:
     (a) Method 1: Try each $n = 1, 2, 3, \ldots$, until you get below the desired error bound.
       i. That is, find $T_1(x)$ and compute the error bound (like discussed above). If the error is below the error given in the problem, then you are done and $n = 1$ works as an answer. If not, find $T_2(x)$ and compute the error bound for $T_2(x)$ (like discussed above). If the error is below the error given in the problem, then you are done and $n = 2$ works as an answer. If not, do the same thing with $T_3(x)$ and $T_4(x)$ and so on.
     (b) Method 2: Manipulate the formula first and see if you can make any conclusions.
       i. The general Taylor inequality is $|f(x) - T_n(x)| \leq \dfrac{M}{(n+1)!}(x-b)^{n+1}$. So you want $\frac{M}{(n+1)!}(x-b)^{n+1} \leq$ 'the given error bound in the given interval'.

ii. So plug in the largest that $x-b$ can be. Also you need to find M (be careful, M may be different for each derivative (remember $|f^{n+1}(x)| \leq M$). Then plug in $n=1$, then $n=2$, then $n=3$, and so on until the value is less than the given error.

Actually, both methods are saying the same thing, but I have explained them in slightly different ways.

7. **Miscellaneous**

- Your life in the first few weeks of this course will be significantly easier if you learn to use the notation properly and if you are organized and complete on your homework. For instance, you should practice using sigma notation (Here is a quick exercise). Each expression below evaluates to a given number, see if you can pair up each value with the correct expression (Solutions are on the last page):

A. $\displaystyle\sum_{k=1}^{4} k$	a. -2
B. $\displaystyle\sum_{i=2}^{3} i^2$	b. -1
C. $\displaystyle\sum_{n=9}^{10} \frac{90}{i}$	c. 0
D. $\displaystyle\sum_{i=1}^{1} \ln(i)$	d. 4
E. $\displaystyle\sum_{k=-1}^{1} (k+1)!$	e. 10
F. $\displaystyle\sum_{n=1}^{3} (-1)^n$	f. 13
G. $\displaystyle\sum_{n=1}^{3} j\cos(\pi j)$	g. 16
H. $\displaystyle\sum_{k=0}^{2} \frac{(2k+1)^2-1}{2}$	h. 19

- You should also take the time to sketch the graphs of the functions so that you can visualize the approximations. If you find the correct $T_1(x)$, $T_2(x)$, *etc*, and you graph them alongside $f(x)$, then you will see that how each approximation corresponds to the original graph. Ultimately, this is what the approximation idea is all about, so it is useful to go to the graphs when you get stuck. Remember you can always generate a graph by plotting a few points and using some Math 124 techniques. I make this last comment because you need to remember that you won't be allowed a graphing calculator when you take exams.

8. **Solutions to the Sigma Notation Exercise** A-e, B-f, C-h, D-c, E-d, F-b, G-a, H-g.