Math 126 - Dr. Loveless

Taylor Notes In Class Example

Let $f(x) = -\ln(1 - \frac{1}{4}x)$

A Find the Taylor series for $f(x)$ based at $b=0$.
Use sigma notation, write out first 3 nonzero terms, & give the interval of convergence.

B What is $T_1(x)$ and $T_2(x)$?
(Try to get these from the series and from using derivatives)

C Use Taylor's Inequality to find a bound on the error for $T_1(x)$ on the interval $I=[-1,1]$.
What's the error for $T_2(x)$ on I?

D Find an interval where the error for $T_1(x)$ is less than 0.01.

E Find n so that the error for $T_n(x)$ is less than 0.0001.
(Find the smallest such n)

F Find a taylor series for $\int_0^x -\ln(1-\frac{1}{4}t)\,dt$

A Recall: $-\ln(1-u) = \sum_{k=1}^{\infty} \frac{1}{k} u^k, \quad -1<u<1$

So $-\ln(1-\frac{1}{4}x) = \sum_{k=1}^{\infty} \frac{1}{k}(\frac{1}{4}x)^k \quad -1<\frac{1}{4}x<1$

$$= \boxed{\sum_{k=1}^{\infty} \frac{1}{k}\frac{1}{4^k} x^k} \quad \boxed{-4<x<4}$$

$$= \frac{1}{4}x + \frac{1}{2}\frac{1}{4^2}x^2 + \frac{1}{3}\frac{1}{4^3}x^3 + \cdots$$

$$= \boxed{\frac{1}{4}x + \frac{1}{32}x^2 + \frac{1}{192}x^3 + \cdots}$$

B From series: $\boxed{\begin{aligned} T_1(x) &= \tfrac{1}{4}x \\ T_2(x) &= \tfrac{1}{4}x + \tfrac{1}{32}x^2 \end{aligned}}$

From derivatives:

$$\left.\begin{aligned} f(x) &= -\ln(1-\tfrac{1}{4}x) \\ f'(x) &= \tfrac{1}{4}\tfrac{1}{(1-\frac{1}{4}x)} \\ f''(x) &= \tfrac{1}{16}\tfrac{1}{(1-\frac{1}{4}x)^2} \end{aligned}\right\} \quad \begin{aligned} T_1(x) &= f(0) + f'(0)x \\ &= -\ln(1) + \tfrac{1}{4}\tfrac{1}{1-\frac{1}{4}(0)}x \\ &= 0 + \tfrac{1}{4}x \end{aligned}$$

$$\boxed{T_1(x) = \tfrac{1}{4}x}$$

$$T_2(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2$$

$$= 0 + \tfrac{1}{4}x + \frac{1}{2}\frac{1}{16}\frac{1}{(1-\frac{1}{4}(0))^2}x^2$$

$$\boxed{T_2(x) = \tfrac{1}{4}x + \tfrac{1}{32}x^2}$$

$\boxed{c}$ $T_1(x) = \frac{1}{4}x$ error on $I=[-1,1]$

(i) $|f''(x)| = \left|\underbrace{\frac{1}{16}\frac{1}{(1-\frac{1}{4}x)^2}}\right|$

Also ~~could say~~ ~~that function~~ also is increasing so max is at $x=1$

only critical value at $x=4$
$\Rightarrow$ max is at one of the endpoints
$f''(-1) = 0.04$
$f''(1) = 0.\overline{1}$ ← max

$\Rightarrow \quad |f''(x)| \le 0.\overline{1} = M$

(ii) So
$$|f(x) - T_1(x)| \le \frac{M}{2!}|x-0|^2 = \frac{0.\overline{1}}{2}|x|^2$$
$$\le \frac{0.\overline{1}}{2}|1|^2 = \boxed{0.0\overline{5}}$$

$T_2(x) = \frac{1}{4}x + \frac{1}{32}x^2$ error on $I=[-1,1]$

(i) $|f'''(x)| = \left|\frac{1}{64}\frac{1}{(1-\frac{1}{4}x)^3}\right|$

similar analysis as above
$\Rightarrow$ max is at $x=1$
$\Rightarrow \quad |f'''(x)| \le \left|\frac{1}{64}\frac{1}{(1-\frac{1}{4}(1))^3}\right| = \underbrace{0.\overline{037}}_{M}$

(ii)
$$|f(x) - T_2(x)| \le \frac{M}{3!}|x-0|^3 = \frac{0.\overline{037}}{3!}|x|^3$$
$$\le \frac{0.\overline{037}}{6}|1|^3 = \boxed{0.00617}$$

$\boxed{D}$ From part $\boxed{C}$, we know the error for $T_1(x)$ on $I=[-1,1]$ is $0.0\overline{5}$ which is too big.

(if we hadn't done part $\boxed{C}$, then you would have needed to guess an interval first and then do part $\boxed{C}$)

So the interval must be smaller than $I=[-1,1]$.

The good news is that our bound

$$|f''(x)| \le 0.\overline{1} = M$$

works on $I=[-1,1]$ so it is valid for any smaller interval.

So we can use this M and solve

$$|f(x)-T_1(x)| \le \frac{M}{2!}|x|^2 \le 0.01$$

we don't know x because we don't know what the endpoints of the new interval will be

This is what we want

$$\frac{0.\overline{1}}{2}|x|^2 \le 0.01$$

$$\Rightarrow |x|^2 \le \frac{0.01}{(0.\overline{1}/2)}$$

$$\Rightarrow |x|^2 \le 0.18$$

$$\Rightarrow |x| \le \sqrt{0.18} = 0.424264$$

$$\Rightarrow -0.424264 < x < 0.424264$$

$I=[-0.424264, 0.424264]$ works

$\boxed{E}$ From $\boxed{C}$,

$$\left.\begin{array}{l} n=1 \Rightarrow \quad T_1(x) \text{ error } \le 0.0\overline{5} \\ n=2 \Rightarrow \quad T_2(x) \text{ error } \le 0.00617 \end{array}\right\} \text{Too BIG}$$

So we continue:

$$n=3 \Rightarrow \quad |f^{(4)}(x)| = \left|\frac{1}{256}\frac{1}{(1-\frac{1}{4}x)^4}\right|$$

$$\le \underbrace{0.012345679}_{\text{since max is at } x=1} = m$$

$$|f(x)-T_3(x)| \le \frac{0.012345679}{4!}|x-0|^4$$

$$\le \frac{0.012345679}{24}|1|^4 = \underbrace{0.000514}_{\text{still too big}}$$

$$n=4 \Rightarrow |f^{(5)}(x)| = \left|\frac{1}{1024}\frac{1}{(1-\frac{1}{4}x)^5}\right|$$

$$\le 0.004115226 = M$$

since max is at $x=1$

$$|f(x)-T_4(x)| \le \frac{0.004115226}{5!}|x-0|^5$$

$$\le \frac{0.004115226}{120}|1|^5 = \boxed{0.00003429}$$

$$\boxed{n=4}$$

$\boxed{F}$ $\displaystyle\int_0^x -\ln(1-\tfrac{1}{4}t)\,dt$ not necessary but useful

$$= \int_0^x \sum_{k=1}^{\infty} \frac{1}{k}\frac{1}{4^k}t^k\,dt = \int_0^x \frac{1}{4}t + \frac{1}{32}t^2 + \frac{1}{192}t^3 + \cdots\,dt$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\frac{1}{4^k}\int_0^x t^k\,dt = \frac{1}{4}\frac{1}{2}t^2 + \frac{1}{32}\frac{1}{3}t^3 + \frac{1}{192}\frac{1}{4}t^4 + \cdots\Big|_0^x$$

$$= \sum_{k=1}^{\infty} \frac{1}{k}\frac{1}{4^k}\frac{1}{k+1}t^{k+1}\Big|_0^x = \frac{1}{8}x^2 + \frac{1}{96}x^3 + \frac{1}{768}\cancel{x}^4 + \cdots$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{1}{k(k+1)4^k}x^{k+1}} = \frac{1}{8}x^2 + \frac{1}{96}x^3 + \frac{1}{768}x^4 + \cdots$$

Open Interval of convergence unchange $\boxed{-4<x<4}$