Taylor Polynomials And Series

- Be able to find Taylor series directly by computing derivatives:

$$f(x) = f(b) + f'(b)(x-b) + \frac{1}{2!}f''(b)(x-b)^2 + \frac{1}{3!}f'''(b)(x-b)^3 + \frac{1}{4!}f^{(4)}(b)(x-b)^4 + \cdots$$

- Taylor Series

You may use the following

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$	$-1 < x < 1$
$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k$	$-1 < x < 1$
$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$	for all x
$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$	for all x
$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$	for all x

Be able to

1. Write any of these in expanded form
2. Find new series by appropriately replacing x
3. Combine these series by adding and subtracting
4. Integrate any of these series term by term.

In generally, be able to work with sigma notation.
For example,

$$\frac{1}{1-3x} - e^x = \sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$
$$= \sum_{n=0}^{\infty} \left(3^n - \frac{1}{n!}\right) x^n$$

And be able to expand with these

5. What is the interval of convergence?

· Taylor Bounds

FOR ANY PROBLEM INVOLVING BOUNDS, ① BOUND THE "NEXT" DERIVATIVE.
② USE TAYLOR'S INEQUALITY.

$\boxed{\text{Type 1: Question}}$ Given $f(x)$, b = base, $I = [c, d]$, n
Find error.

ⓐ Compute $f^{(n+1)}(x)$

ⓑ Find M = maximum of $f^{(n+1)}(x)$ on $I = [c, d]$

$|f^{(n+1)}(x)| \leq M$

often this can be done by simply inspecting $y = f^{(n+1)}(x)$

However, to maximize in general
(i) Compute y' and find critical points
(ii) M occurs at a critical point or endpoint

ⓒ

$$\text{error} = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1} \leq \underbrace{\frac{M}{(n+1)!}|d-b|^{n+1}}_{\text{this is a number}}$$

$\boxed{\text{Type 2 Question}}$ Given $f(x)$, b = base, $I = [c, d]$, error
Find n

ⓐ $n=1$: Find M_1 $\quad |f''(x)| \leq M_1$

compute $|f(x) - T_1(x)| \leq \frac{M_1}{2!}|x-b|^2 = \underbrace{\frac{M_1}{2}|d-b|^2}_{\text{is this small enough?}}$

if yes, STOP
if no, try $n = 2$

ⓑ $n=2$: Find M_2 $\quad |f^{(3)}(x)| \leq M_2$

compute $|f(x) - T_2(x)| \leq \frac{M_2}{3!}|x-b|^3 \leq \underbrace{\frac{M_2}{3!}|d-b|^3}_{\text{is this small enough?}}$

if yes, STOP
if no, try $n = 3$

AND SO FORTH

Type 3 Question Given $f(x)$, b = base, n, error
Find $I = [c, d]$

ⓐ "Guess" $I = [c_0, d_0]$

compute $|f^{(n+1)}(x)| \leq M_n$ on this interval

$$|f(x) - T_n(x)| \leq \frac{M_n}{(n+1)!}|x-b|^{n+1} \leq \underbrace{\frac{M_n}{(n+1)!}|d_0-b|^{n+1}}$$

is this small enough?
if yes, take $c = c_0$, $d = d_0$
If no, move on

ⓑ If your error is too big, then the interval is too big.

Since $|f^{(n+1)}(x)| \leq M_n$ on $I = [c_0, d_0]$
then $|f^{(n+1)}(x)| \leq M_n$ on any smaller interval. (THIS IS ONLY VALID BECAUSE n HAS NOT CHANGE!)

Since M_n, n & b are all known we can solve

$$\frac{M_n}{(n+1)!}|x-b|^{n+1} \leq \text{ERROR}$$

$$|x-b|^{n+1} \leq \text{ERROR}\,\frac{(n+1)!}{M_n}$$

$$|x-b| \leq \underbrace{\left(\text{ERROR}\,\frac{(n+1)!}{M_n}\right)^{1/n}}_{C}$$

$$-C \leq x-b \leq C$$

$$\boxed{b - C \leq x \leq b + C}$$