1. (12 pts) The acceleration of a particle is given by $\mathbf{a}(t) = \langle 0, 3\sin(t), 3\cos(t)\rangle$. In addition, the initial velocity and position are given by $\mathbf{v}(0) = \langle 1, -3, 0\rangle$ and $\mathbf{r}(0) = \langle 3, 2, 1\rangle$.

(a) Find the position vector function, $\mathbf{r}(t)$. (Please double-check your initial conditions).

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle c_1, -3\cos(t) + c_2, 3\sin(t) + c_3 \rangle$$

$$c_1 = 1, \quad -3 + c_2 = -3, \quad 0 + c_3 = 0$$

$$\Rightarrow \vec{v}(t) = \langle 1, -3\cos(t), 3\sin(t) \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle t + d_1, -3\sin(t) + d_2, -3\cos(t) + d_3 \rangle$$

$$d_1 = 3, \quad 0 + d_2 = 2, \quad -3 + d_3 = 1$$

$$\Rightarrow \boxed{\vec{r}(t) = \langle t + 3, -3\sin(t) + 2, -3\cos(t) + 4 \rangle}$$

(b) For $\mathbf{r}(t)$ above, find the unit tangent, $\mathbf{T}(t)$, and the principal unit normal, $\mathbf{N}(t)$, at time t.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 1, -3\cos(t), 3\sin(t) \rangle}{\sqrt{1 + 9\cos^2(t) + 9\sin^2(t)}} = \boxed{\left\langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\cos(t), \frac{3}{\sqrt{10}}\sin(t) \right\rangle}$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \frac{\left\langle 0, \frac{3}{\sqrt{10}}\sin(t), \frac{3}{\sqrt{10}}\cos(t) \right\rangle}{\sqrt{0 + \frac{9}{10}\sin^2(t) + \frac{9}{10}\cos^2(t)}} = \boxed{\langle 0, \sin(t), \cos(t) \rangle}$$

2. (10 pts) Let $f(x,y) = 4xy - 3y + \frac{1}{x} - \frac{1}{4}\ln(y)$. Find and classify all the critical points of $f(x,y)$. **Clearly** show your work in using the second derivative test. (Put a box around your critical points and clearly write the words 'local max', 'local min' or 'saddle point' appropriately next to each point).

$$f_x = 4y - \frac{1}{x^2} \overset{?}{=} 0 \Rightarrow y = \frac{1}{4x^2}$$

$$f_y = 4x - 3 - \frac{1}{4y} \overset{?}{=} 0 \Rightarrow 4x - 3 - x^2 \overset{?}{=} 0$$

$$\Rightarrow 0 = x^2 - 4x + 3$$

So $(x-3)(x-1) = 0$

$x = 1 \Rightarrow y = 1/4$

or

$x = 3 \Rightarrow y = \frac{1}{36}$

$$f_{xx} = \frac{2}{x^3}, \quad f_{yy} = \frac{1}{4y^2}, \quad f_{xy} = 4$$

$\boxed{\text{AT } (1, 1/4) \quad \text{SADDLE POINT}}$

$$\underbrace{f_{xx} = 2, \; f_{yy} = 4, \; f_{xy} = 4}$$

$$D = (2)(4) - (4)^2 = -8 < 0$$

$\boxed{\text{AT } (3, \tfrac{1}{36}) \quad \text{LOCAL MINIMUM}}$

$$\underbrace{f_{xx} = \frac{2}{27}, \quad f_{yy} = 324, \; f_{xy} = 4}$$

$$\begin{cases} D = \left(\frac{2}{27}\right)(324) - (4)^2 = 8 > 0 \\ \text{AND } \; f_{xx} = \frac{2}{27} > 0 \end{cases}$$

3. (14 pts) **The two problems below are not related. Simplify your answer in <u>exact form</u>.**

(a) Find the linear approximation, $L(x,y)$, to $z^3+e^{3y}=1+x^4+z\sin(y)$ at $(x,y,z)=(-1,0,1)$. (Hint: First, use implicit differentiation to find $\frac{dz}{dx}$ and $\frac{dz}{dy}$).

$\frac{\partial z}{\partial x}$: $3z^2\frac{\partial z}{\partial x}+0=0+4x^3+\frac{\partial z}{\partial x}\sin(y)$

at $(-1,0,1) \Rightarrow 3\frac{\partial z}{\partial x}=-4+0 \Rightarrow \frac{\partial z}{\partial x}=-4/3$

$\frac{\partial z}{\partial y}$: $3z^2\frac{\partial z}{\partial y}+3e^{3y}=0+0+\frac{\partial z}{\partial y}\sin(y)+z\cos(y)$

at $(-1,0,1) \Rightarrow 3\frac{\partial z}{\partial y}+3=0+1 \Rightarrow \frac{\partial z}{\partial y}=-2/3$

ASIDE:

$$\frac{\partial z}{\partial x}=\frac{4x^3}{3z^2-\sin(y)}$$

$$\frac{\partial z}{\partial y}=\frac{-3e^{3y}+z\cos(y)}{3z^2-\sin(y)}$$

$$\boxed{L(x,y)=1-\frac{4}{3}(x+1)-\frac{2}{3}y}$$

(b) Let D be the region in the first quadrant of the xy-plane bounded by $y=2x-1$ and $y^2=x$ (as shown). Evaluate $\iint_D 4x\,dA$.

LEFT: $x=y^2$

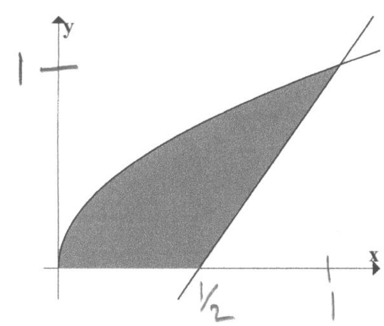
RIGHT: $x=\frac{y+1}{2}$

$y^2=\frac{y+1}{2}$

$\Rightarrow 2y^2-y-1=0$

$(2y+1)(y-1)=0$

$y=1$



$$\int_0^1\int_{y^2}^{\frac{y+1}{2}} 4x\,dx\,dy$$

$$=\int_0^1 2x^2\Big|_{y^2}^{\frac{y+1}{2}}dy$$

$$=\int_0^1 2\frac{(y+1)^2}{4}-2y^4\,dy$$

$$=\frac{(y+1)^3}{6}-\frac{2}{5}y^5\Big|_0^1$$

$$=\left(\frac{8}{6}-\frac{2}{5}\right)-\left(\frac{1}{6}-0\right)$$

$$=\frac{7}{6}-\frac{2}{5}=\frac{35}{30}-\frac{12}{30}=\boxed{\frac{23}{30}}$$

HARD WAY:

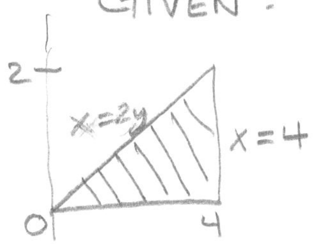
$$\int_0^{\frac{1}{2}}\int_0^{\sqrt{x}}4x\,dy\,dx+\int_{\frac{1}{2}}^1\int_{2x-1}^{\sqrt{x}}4x\,dy\,dx$$

4. (14 pts) **The two problems below are not related. Simplify your answer in exact form.**

(a) Reverse the order of integration and evaluate $\displaystyle\int_0^2 \int_{2y}^4 8\sqrt{x^2+1}\,dx\,dy$.

GIVEN: LEFT $x=2y$
RIGHT $x=4$

BOT: $y=0$
TOP: $y=\frac{1}{2}x$



$$= \int_0^4 \int_0^{\frac{1}{2}x} 8\sqrt{x^2+1}\,dy\,dx$$

$$= \int_0^4 8\sqrt{x^2+1}\, y\Big|_0^{\frac{1}{2}x}\,dx$$

$$= \int_0^4 4\sqrt{x^2+1}\, x\,dx \qquad u=x^2+1,\ du=2x\,dx$$

$$= \int_1^{17} 2\sqrt{u}\,du$$

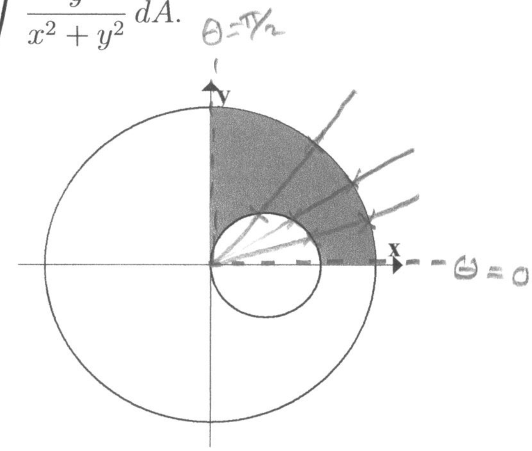
$$= \frac{4}{3}u^{3/2}\Big|_1^{17}$$

$$= \boxed{\frac{4}{3}(17)^{3/2} - \frac{4}{3}}$$

(b) Let R be the region in the first quadrant between the circle $x^2+y^2=9$ and the circle $x^2+y^2=2x$ (as shown). Using polar coordinates, evaluate $\displaystyle\iint_R \frac{y}{x^2+y^2}\,dA$.

OUTSIDE: $x^2+y^2=9 \Rightarrow r=3$

INSIDE: $x^2+y^2=2x \Rightarrow r^2=2r\cos\theta$
$\Rightarrow r=2\cos\theta$



$$\int_0^{\pi/2}\int_{2\cos\theta}^{3} \frac{r\sin\theta}{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left(r\Big|_{2\cos\theta}^{3} \right) d\theta$$

$$= \int_0^{\pi/2} \sin\theta\,(3-2\cos\theta)\,d\theta \qquad u=3-2\cos\theta,\ du=2\sin\theta\,d\theta$$

$$= \frac{1}{2}\int_1^3 u\,du$$

$$= \frac{1}{4}u^2\Big|_1^3$$

$$= \frac{1}{4}(9-1) = \boxed{2}$$