13.2 and 13.3 Computation Practice

Recall:

- $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ = the tangent vector
- $|\mathbf{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ = speed
- $\mathbf{r}''(t) = \langle x''(t), y''(t), z''(t)\rangle$ = acceleration vector
- $\mathbf{T}(t) = \dfrac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t)$ = the unit tangent vector
- $\mathbf{N}(t) = \dfrac{1}{|\mathbf{T}'(t)|}\mathbf{T}'(t)$ = principal unit normal vector
- $\int \mathbf{r}(t)\,dt = \langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt\rangle$
- The tangent line to $\mathbf{r}(t)$ at $t = t_0$ is given by
  $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$,
  where $\langle x_0, y_0, z_0\rangle = \mathbf{r}(t_0)$, and $\langle a, b, c\rangle = \mathbf{r}'(t_0)$.
- $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$ = Arc Length
- $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ = curvature.

You try:
Consider the vector function $\mathbf{r}(t) = \langle t, \cos(2t), \sin(2t)\rangle$.

1. Find $\mathbf{r}'(t)$, $\mathbf{T}(t)$, $\mathbf{r}''(t)$, and $\mathbf{N}(t)$.
2. Find $\int \mathbf{r}(t)\,dt$.
3. Find the equation for the tangent line at $t = \frac{\pi}{4}$.
4. Find the arc length from 0 to 3.
5. Reparameterize in terms of arc length.
6. Find the curvature at $t = 0$.

Solutions on the next page

Solutions:

1. $\mathbf{r}'(t) = \langle 1, -2\sin(2t), 2\cos(2t)\rangle$.
   $\mathbf{T}(t) = \frac{1}{\sqrt{1+4\sin^2(2t)+4\cos^2(2t)}}\langle 1, -2\sin(2t), 2\cos(2t)\rangle = \langle \frac{1}{\sqrt{5}}, -\frac{2\sin(2t)}{\sqrt{5}}, \frac{2\cos(2t)}{\sqrt{5}}\rangle$.
   $\mathbf{r}''(t) = \langle 0, -4\cos(2t), -4\sin(2t)\rangle$.
   $\mathbf{T}'(t) = \langle 0, -\frac{4}{\sqrt{5}}\cos(2t), -\frac{4}{\sqrt{5}}\sin(2t)\rangle$.
   $\mathbf{N}(t) = \langle 0, -\cos(2t), -\sin(2t)\rangle$.

2. $\displaystyle\int \mathbf{r}(t)\,dt = \left\langle \frac{1}{2}t^2 + C_1, \frac{1}{2}\sin(2t) + C_2, -\frac{1}{2}\cos(2t) + C_3 \right\rangle$.

3. $\mathbf{r}(\pi/4) = \langle \pi/4, 0, 1\rangle$.
   $\mathbf{r}'(\pi/4) = \langle 1, -2, 0\rangle$.
   Thus, $x = \pi/4 + t$, $y = 0 - 2t$, $z = 1$.

4. $\displaystyle\int_0^3 \sqrt{1 + 4\sin^2(2t) + 4\cos^2(2t)}\,dt = \int_0^3 \sqrt{5}\,dt = 3\sqrt{5}$

5. $\displaystyle s(t) = \int_0^t \sqrt{1 + 4\sin^2(2u) + 4\cos^2(2u)}\,du = t\sqrt{5}$, so $t = s/\sqrt{5}$.
   Thus, $\mathbf{r}(s) = \langle s/\sqrt{5}, \cos(2s/\sqrt{5}), \sin(2s/\sqrt{5})\rangle$

6. $\mathbf{r}'(0) = \langle 1, 0, 2\rangle$.
   $\mathbf{r}''(0) = \langle 0, -4, 0\rangle$.
   $|\mathbf{r}'(0)| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{5}$.
   $\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle 8, 0, -4\rangle$.
   Thus, $\kappa(0) = \frac{|\mathbf{r}'(0)\times\mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{\sqrt{8^2+0^2+4^2}}{\sqrt{5}^3} = \frac{\sqrt{80}}{\sqrt{5}^3} = \frac{4\sqrt{5}}{\sqrt{5}^3} = \frac{4}{5}$.