Worksheet 3 Solutions

1. *Surfaces:* For the surface $x + y^2 - z^2 = 4$.
   - If $x = k$ is fixed, then the traces are **hyperbolas**.
   - If $y = k$ is fixed, then the traces are **parabolas**.
   - If $z = k$ is fixed, then the traces are **parabolas**.
   - This shape is called a **hyperbolic paraboloid**.

2. *Basic Parametric:* For $x = t$, $y = t\sin(\pi t)$, $z = t\cos(\pi t)$, describe the surface of motion.
   Answer: Since $x = t$, we can say the motion occurs on the intersection of the surfaces $y = x\sin(\pi x)$ and $z = x\cos(\pi x)$. Squaring and summing y and z gives $y^2 + z^2 = x^2\sin^2(\pi x) + x^2\cos^2(\pi x) = x^2$. Thus, the motion occurs on the surfaces $y^2 + z^2 = x^2$. This surface is called a **cone**.

3. *Basic Parametric:* Assume $\mathbf{r}_1(t) = \langle t, 5t, t^2\rangle$ and $\mathbf{r}_2(t) = \langle 5 - t, 7t + 1, t^3 + 1\rangle$.
   - Find all points at which their **paths** intersect.
     Answer: This question is asking if the same (x, y, z) points ever occur (not necessarily with the same parameter choice). So we need to use different symbols for each parameter and then set the x, y and z equations equal.

     $$\begin{array}{lll} (1) & t & = 5 - u \\ (2) & 5t & = 7u + 1 \\ (3) & t^2 & = u^3 + 1 \end{array}$$

     Combining conditions (1) and (2) gives $5(5 - u) = 7u + 1$, which simplifies to $24 = 12u$. Thus, the only way x- and y-coordinates can be the same is if $u = 2$ and $t = 5 - 2 = 3$. (Note that you can check your work, $u = 2$ and $t = 3$ both give $x = 3$ and $y = 15$).

     And the last step is to see if the z-coordinates are the same for these parameter values. For these values, we get $t^2 = 9$ and $u^3 + 1 = 9$, so YES, the curves do intersect.

     The one point of intersection of the two curves if $(x, y, z) = (3, 15, 9)$.
   - Do the object every **collide**? (If so, find the time when they collide. If not, explain why.)
     Answer: If you use the given parameterizations and assume that the parameters are actually time, then this question is asking if the two objects are ever at the same (x, y, z) point at the exact same time t. No additional work is needed to solve this problem. We just saw in the last part that the only way the (x, y, z) point can be the same is when the first time is $t = 3$ and when the second time is $t = 2$. So the answer is NO, the objects do not collide.

4. *Parametric Calculus:* For the curve given by $\mathbf{r}(t) = \langle t^2 + 1, t^3, 1 - 5t\rangle$.
   - *Answer:* $\mathbf{r}'(t) = \langle 2t, 3t^2, -5\rangle$ = the derivative vector (a tangent vector).
   - *Answer:* $\mathbf{r}'(1) = \langle 2, 3, -5\rangle$ and $\mathbf{T}(1) = \frac{1}{\sqrt{4+9+25}}\langle 2, 3, -5\rangle = \left\langle \frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, -\frac{5}{\sqrt{38}} \right\rangle$.
   - *Answer:* $x = 2 + 2t$, $y = 1 + 3t$, $z = -4 - 5t$ because $\mathbf{r}(1) = \langle 2, 1, -4\rangle$ and $\mathbf{r}'(1) = \langle 2, 3, -5\rangle$.

- *Challenge Answer:* The curve intersects the xy-plane when $z = 0$, so $0 = z = 1 - 5t$. Thus, $t = 1/5$ at this intersection.

  The tangent direction of the curve at $t = 1/5$ is $\mathbf{r}'(1/5) = \langle 2/5, 3/25, -5\rangle$. We want to know the angle this makes with the ground.

  Option 1: The vector directly above it on the ground is $\langle 2/5, 3/25, 0\rangle$. So find the angle between $\mathbf{a} = \langle 2/5, 3/25, 0\rangle$ and $\mathbf{b} = \langle 2/5, 3/25, -5\rangle$ using $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$ which gives

  $$\cos^{-1}\left(\frac{(2/5)^2 + (3/25)^2 + 0}{\sqrt{(2/5)^2 + (3/25)^2}\sqrt{(2/5)^2 + (3/25)^2 + (-5)^2}}\right) = 85.23 \text{ degrees} .$$

  Option 2: A downward normal vector for the ground is $\langle 0, 0, -1\rangle$. Note: If θ is the desired angle between $\langle 2/5, 3/25, -5\rangle$ and the ground and α is the angle between $\langle 0, 0, -1\rangle$ and $\langle 2/5, 3/25, -5\rangle$, then $\alpha + \theta = 90$ degrees. So let's find α and subtract it from 90 degrees. The angle between $\mathbf{a} = \langle 0, 0, -1\rangle$ and $\mathbf{b} = \langle 2/5, 3/25, -5\rangle$ is

  $$\alpha = \cos^{-1}\left(\frac{5}{\sqrt{(-1)^2}\sqrt{(2/5)^2 + (3/25)^2 + (-5)^2}}\right) = 4.77 \text{ degrees} ,$$

  Thus, $\theta = 90 - 4.77 = 85.23$ degrees.