Taylor Polynomials and Series Review

Taylor polynomials: I give the 1st, 2nd, 3rd and, general, nth Taylor polynomials below.

$$T_1(x) = \sum_{k=0}^{1} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b).$$

$$T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2.$$

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3.$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k \quad = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \cdots + \frac{f^{(n)}(b)}{n!}(x-b)^n.$$

Taylor inequalities: I give the first error bound and the general error bound below.

$$\text{ERROR} = |f(x) - T_1(x)| \leq \frac{M}{2!}|x-b|^2$$, where $|f''(x)| \leq M$ on the interval, and in general,

$$\text{ERROR} = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1}$$, where $|f^{(n+1)}(x)| \leq M$ on the interval.

Three types of error questions:

Given an interval $[b-a, b+a]$, find the $T_n(x)$ error bound:

1. Find $|f^{(n+1)}(x)|$.
2. Determine a bound (a 'tight' bound if easy to find, but any bound will do) for $|f^{(n+1)}(x)| \leq M$ on the interval.
3. In Taylor's inequality $\frac{M}{(n+1)!}|x-b|^{n+1}$ replace M and replace x by an endpoint.

Find an interval so that $T_n(x)$ has a desired error:

1. Write $[b-a, b+a]$ and you will solve for a.
2. Find $|f^{(n+1)}(x)|$.
3. Determine a bound (the maximum value if possible) for $|f^{(n+1)}(x)| \leq M$ on the interval, this may involve the symbol a.
4. In Taylor's inequality $\frac{M}{(n+1)!}|x-b|^{n+1}$ replace M and replace x by an endpoint (this will involve the symbol a).
5. Then solve for a to get the desired error.

Given an interval $[b-a, b+a]$, find n so that $T_n(x)$ gives a desired error:
(There is no good general way to solve for the answer in this case, you just use trial and error).

1. Find the error for $n = 1$, then $n = 2$, then $n = 3$, etc. Once you get an error less than the desired error, you stop.
2. If you spot a pattern in the errors, then use the pattern to solve for the first time the error will be less than the desired error.

Taylor series: Know these four series very well and know their intervals of convergence.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \qquad = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \qquad \text{, for all } x.$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \qquad = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots \qquad \text{, for all } x.$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \qquad = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \cdots \qquad \text{, for all } x.$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad = 1 + x + x^2 + x^3 + \cdots \qquad \text{, for } -1 < x < 1.$$

Substituting into series (examples): Know how to substitute into known series, including the interval of convergence.

$$e^{2x^3} = \sum_{k=0}^{\infty} \frac{1}{k!} 2^k x^{3k} \qquad = 1 + 2x^3 + \tfrac{2^2}{2!}x^6 + \tfrac{2^3}{3!}x^9 + \cdots \qquad \text{, for all } x.$$

$$\sin(5x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} 5^{2k+1} x^{2k+1} \qquad = 5x - \tfrac{5^3}{3!}x^3 + \tfrac{5^5}{5!}x^5 + \cdots \qquad \text{, for all } x.$$

$$\cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k} \qquad = 1 - \tfrac{1}{2!}x^4 + \tfrac{1}{4!}x^8 + \cdots \qquad \text{, for all } x.$$

$$\frac{1}{1+3x} = \sum_{k=0}^{\infty} (-3)^k x^k \qquad = 1 - 3x + 3^2x^2 - 3^3x^3 + \cdots \qquad \text{, for } -1 < -3x < 1.$$

Multiplying out (examples): Know how to mulitply/divide a given series by x or x^2 or x^3, etc.

$$x^3 e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+3} \qquad = x^3 + x^4 + \tfrac{1}{2!}x^5 + \tfrac{1}{3!}x^6 + \cdots \qquad \text{, for all } x.$$

$$\frac{x^2}{1+2x} = \sum_{k=0}^{\infty} (-2)^k x^{k+2} \qquad = x^2 - 2x^3 + 2^2x^4 - 2^3x^5 + \cdots \qquad \text{, for } -1 < 2x < 1.$$

Integrating/Differentiating (examples): Know how to integrate and differentiate.

$$-\ln(1-x) = \int_0^x \frac{1}{1-t}\,dt = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} \qquad = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots \qquad \text{, for } -1 < x < 1.$$

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \qquad = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \cdots \qquad \text{, for } -1 < x < 1.$$

$$\int e^{x^3}\,dx = C + \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{3k+1} x^{3k+1} \qquad = C + x + \tfrac{1}{2!(4)}x^4 + \tfrac{1}{3!(7)}x^7 + \cdots \qquad \text{, for all } x.$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} k x^{k-1} \qquad = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad \text{, for } -1 < x < 1$$