Taylor Polynomials, Taylor Series, and Final Review

This worksheet is designed to help you to start thinking about the final and to help sort out your understanding of Taylor polynomials. **In small groups discuss the first four questions.** Hand in your work for the first four questions to get credit for your worksheet, then keep this handout for studying. If time permits, discuss questions 5, 6, and 7 with your TA. Most of these questions are from old midterms and final exams. This is not a comprehensive list of topics and this should not be your only source of studying, I just wanted to give you a few old exam questions to start your studying.

1. Find the 2nd-degree Taylor polynomial, $T_2(x)$ for the function $f(x) = \ln(\ln x)$ based at $x = e$.

2. Consider the function $f(x) = \sin\left(\frac{\pi x}{6}\right)$.

   (a) Find $T_2(x)$, the second order Taylor polynomial for $f(x)$ centered at $a = 1$.

   (b) Use Taylor's inequality to find an upper bound on $|f(1.1) - T_2(1.1)|$.

3. (a) Find the quadratic approximation, $T_2(x)$, based at $b = 1$, for the function

$$f(x) = x \ln x.$$

   (b) Use $T_2(x)$ to approximate $0.9 \ln(0.9)$.

   (c) Using Taylor's inequality, estimate the error of the approximation you obtained in (b).

4. Consider the function $f(x) = x^3 + x$.

   (a) Find the second Taylor polynomial T_2 of f based at $b = 1$.

   (b) Use Taylor's inequality to find an interval J around b such that the error $|T_2(x) - f(x)|$ is less than 0.001 for all x in J.

5. Approximate the integral

$$\int_0^2 \sin(x^2)\, dx$$

   by using the first four non-zero terms of a Taylor series. Given a decimal approximation of your result.

6. Write out the first four terms of the Taylor series for the function $f(x) = \dfrac{1}{1+5x} + \dfrac{1}{3+x}$.

7. Give the coefficient on x^{11} in the Taylor series for $f(x) = x^3 e^{x^2}$ based at $b = 0$.

8. Let $f(x) = x^3 \cos(5x^2)$. Write down the Taylor series about $a = 0$ for the indefinite integral $\int f(x)\, dx$.

9. Consider the function $f(x) = \ln(3 + 2x^2)$.

   (a) Compute $f'(x)$ and find its Taylor series centered at zero.

   (b) Use part (a) to find the Taylor series centered at zero for $f(x)$. (**Hint:** What is $f(0)$?)

   (c) What is the radius of convergence of the series you found in part (b)?

10. Find a vector $\mathbf{v}$ which satisfies both of the following conditions:

    (i) $\mathbf{v}$ is orthogonal to $\langle 2, 1, 4\rangle$,

    (ii) the cross product of $\mathbf{v}$ and $\langle 1, 2, 0\rangle$ equals $\langle 2, -1, 0\rangle$.

11. Let L_1 be the line given by the parametric equations

$$x = 2t, y = 0, z = 4 - 4t,$$

and let L_2 be the line given by the parametric equations

$$x = 2 - 2u, y = 3u, z = 0.$$

    (a) Find the point of intersection of L_1 and L_2.

    (b) Find an equation of the plane that contains both L_1 and L_2. Give your answer in the form $ax + by + cz = d$.

12. Find the parametric equations for the line that is the intersection of the plane

$$x + y + 2z = 1$$

and the plane

$$3x - y + 4z = 1.$$

13. Find an equation for the plane through the origin that is perpendicular to the planes $5x-y+z = 1$ and $2x + 2y - 3z = 2$.

14. Suppose the trajectory of a particle is given by

$$\mathbf{r}(t) = \sin t\ \mathbf{i} + \cos t\ \mathbf{j} + t\ \mathbf{k}.$$

Calculate the magnitude of the normal component of the acceleration experienced by the particle at $t = 1$.

15. Consider the space curve represented by the vector function $\mathbf{r}(t) = \langle \cos(t), \cos(t), \sqrt{2}\sin(t)\rangle$, where $0 \le t \le 2\pi$.

    (a) Compute $\mathbf{r}'(t)$.

    (b) Reparametrize the curve with respect to the arclength.

    (c) Let $P = (1/2,\ 1/2,\ \sqrt{3/2})$. Find the following.

        i. A parametrization of the tangent line for the curve at P.

        ii. The curvature of the curve at P.

        iii. An equation of the osculating plane for the curve at P.

        iv. An equation of the normal plane for the curve at P.

16. The position function of a spaceship is $\mathbf{r}(t) = \langle 3 + t, 2 + \ln t, 7 + t^2\rangle$ and the coordinates of the space station are $(7, 5, 14)$. The captain wants the spaceship to coast into the space station. When should the engines be turned off? (That is, we want the value of t for which the tangent line with intersect (7,5,14)).

17. Find the vector function $\mathbf{r}(t)$ such that the acceleration is $\mathbf{a}(t) = \mathbf{i} - 12t^2\mathbf{j} + 2t\mathbf{k}$ and the initial position and velocity are given by $\mathbf{r}(0) = \mathbf{i} + \mathbf{k}$ and $\mathbf{v}(0) = 2\mathbf{j}$.

18. Consider the function $f(x,y) = e^{3x+5y-1}$.

   (a) Sketch the level sets of f, $f(x,y) = k$, for $k = e^{-1}$ and $k = 1$. What are the level sets if $k \leq 0$?

   (b) Calculate the partial derivatives f_x and f_y.

   (c) Write an equation for the tangent plane to the graph of $f(x,y)$ at the point $(2,-1,1)$.

   (d) Use the linear approximation for f at $(2,-1)$ to estimate the value $f(1.8,-0.9)$.

19. Find and classify all the critical points of the function $f(x,y) = x^3 + y^2 + 2xy$.

20. Integrate the function $f(x,y) = x + y$ over the region bounded by $x + y = 2$ and $y^2 - 2y - x = 0$.

21. (a) Reverse the order of integration for the integral

$$\int_0^4 \int_{\sqrt{x}}^2 xy \, dy \, dx \, .$$

   (b) Evaluate the integral in part (a). You may integrate either in the original order or in the reversed order.

22. Evaluate the following iterated integral:

$$\int_0^3 \int_0^{9-x^2} \frac{xe^{3y}}{9-y} \, dy \, dx$$

23. Find the volume of the region between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and bounded above by $z = y^2$ and bounded below by $z = 0$.