13.3, 13.4, and 14.1 Review

This review sheet discusses, in a very basic way, the key concepts from these sections. This review is not meant to be all inclusive, but hopefully it reminds you of some of the basics. Please notify me if you find any typos in this review.

1. **13.3 Arc Length and Curvature**

   (a) *Arc Length*: If a space curve has the vector equation $\mathbf{r}(t) =< f(t), g(t), h(t) >$ and the curve is traversed exactly once from $t = a$ to $t = b$, then

   $$\text{ARC LENGTH } = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

   (b) *Arc Length Parametrization*: Occasionally, we want to know the location in terms of the arc length. To illustrate this point consider the following questions:

      i. "Where is an airplane located in 10 minutes?" This questions is asked in terms of time. So a parametric equation in terms of time, t, would be useful to answer the question.

      ii. "Where is an airplane located after traveling 20 miles?" This question is asked in terms of arc length. So a parametric equation in terms of arc length, s, would be useful.

   How to Reparametrize in Terms of Arc Length

      i. Compute the arc length function from the given starting time, $t = a$:

      $$s(t) = \int_a^t |\mathbf{r}'(u)|\, du$$

      ii. Solve for t in terms of s.

      iii. Rewrite your function in terms of s.

   (c) *Curvature*: The curvature measures how quickly the direction of the tangent vector is changing with respect to arc length. To make sure that the length of the tangent vector does not effect our computation, we start with the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. Thus, the **curvature** of a curve is

   $$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \text{'magnitude of the rate of change of the tangent vector with respect to arc length'}$$

   There are several equivalent ways to write this. Here are some variants:

   $$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

   $$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

   $$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} \qquad \text{(for 2 dimensions)}$$

   (d) *Normal, Unit Normal, and Binormal Vectors* We already are able to find a vector that is tangent to our plane by using the derivative. That is, $\mathbf{r}'(t)$ = 'a tangent vector' and $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ = 'the unit tangent vector'.

   Now we used $\mathbf{T}(t)$ to find other vectors that give more information about the curve. Here is a collection of facts about other vectors that we can find:

$\mathbf{T}'(t) =$ 'a normal vector (a vector orthogonal to $\mathbf{T}(t)$)'

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \text{ 'the principal unit normal'}$$

$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) =$ 'the binormal vector (orthogonal to the tangent and unit normal)'

That is, $\mathbf{T}'(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ are all vectors that are perpendicular to the curve at the given value of t.

We can use this these vectors to find various planes. In this section, you only need to know about **normal planes**. The normal plane is determined by the vectors $\mathbf{N}(t)$ and $\mathbf{B}(t)$. That is, you can use $\mathbf{n} = \mathbf{T(t)}$ as your normal vector to the plane and $\mathbf{r}_0 =< x_0, y_0, z_0 >$, where (x_0, y_0, z_0) is the point of your tangent, to find the equation for the normal plane. The normal plane slices through the space curve orthogonal to the tangent point.

2. **13.4 Motion in Space**: Here we introduce in a basic way how derivatives and integrals of vector functions can be used to answer questions about position, velocity and acceleration in 3 dimensions.

   (a) First, we have
   $\mathbf{v}(t) = \mathbf{r}'(t) =$ 'the velocity vector'
   $|\mathbf{v}(t)| = |\mathbf{r}'(t)| =$ 'the speed function'
   $\mathbf{a}(t) = \mathbf{r}''(t) =$ 'the acceleration vector'

   (b) From Newton's Second Law of Motion, we have $\mathbf{F}(t) = m\mathbf{a}(t)$. (That is, the force vector is equal to mass times the acceleration vector). This relationship is needed for projectile problems. Please read examples 4, 5, and 6 of the text for some uses of this law.

   (c) The accelaration can be decomposed into two parts: a tangential component and a normal component. A description of their interpretation is given in the middle of page 875 of your text. To compute them, you can use the following:

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \qquad \text{(tangential component)}$$

$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} \qquad \text{(normal component)}$$

3. **14.1 Functions of Several Variables**: Chapter 13 was focused on calculus for 3 dimensional **curves**. In Chapter 14, we will apply calculus to 3 dimensional **surfaces**. We already introduced 3 dimensional surfaces in section 12.6. However, now we will focus on functions of the form

$$z = f(x, y).$$

   We can think as z as the height of the surface above the xy-plane.

   (a) *Basics*: If one of the variables is fixed, then the function will have only two variables. For example, $z = f(4, y)$ is the function where $x = 4$ is fixed and y is allowed to change. That is, it is a function of y. As another example, $z = f(x, -7)$ is a function of x.

   (b) *Domain*: The domain is all allowable input values. The domain for a function of the form $z = f(x, y)$ is some set of (x, y) values that can be sketched in 2 dimensions.

Recall the following domain restriction rules:

FUNCTION	RESTRICTION
$z = \sqrt{\text{BLAH}}$	$\text{BLAH} \geq 0$
$z = \dfrac{1}{\text{BLAH}}$	$\text{BLAH} \neq 0$
$z = \ln(\text{BLAH})$	$\text{BLAH} > 0$
$z = \sin^{-1}(\text{BLAH})$	$-1 \leq \text{BLAH} \leq 1$
$z = \cos^{-1}(\text{BLAH})$	$-1 \leq \text{BLAH} \leq 1$

(c) *Graphing and Level Curves*: For 3 dimensional surfaces of the form $z = f(x, y)$, we use the same techniques as in section 12.6. Recall, one of our main techniques was *traces* (we fixed on variable and graded in 2 dimensions).

When we fix z at different values and graph the corresponding 2 dimensional curves on the same set of axes, we call these *level curves*.

For example: if $z = x^2 - y$, then we graph the level curves as follows:

i. **Fix several different values of z.** For this example, let's try $z = -1$, $z = 0$, $z = 1$ and $z = 2$.

ii. **For each z, graph the 2D curve.**

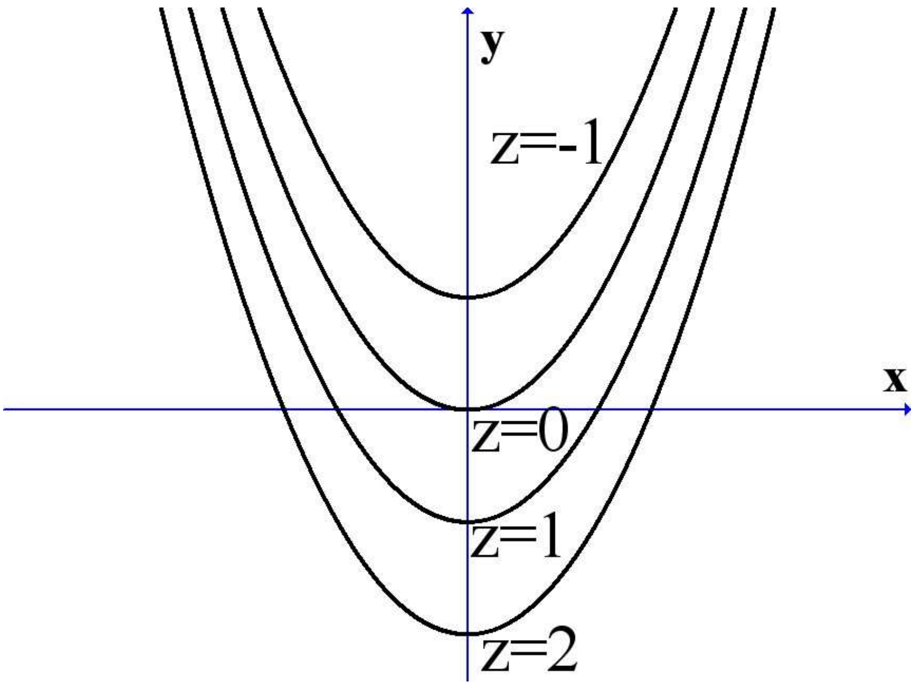
$$z = -1 \Longrightarrow -1 = x^2 - y \Longrightarrow y = x^2 + 1$$

$$z = 0 \Longrightarrow 0 = x^2 - y \Longrightarrow y = x^2$$

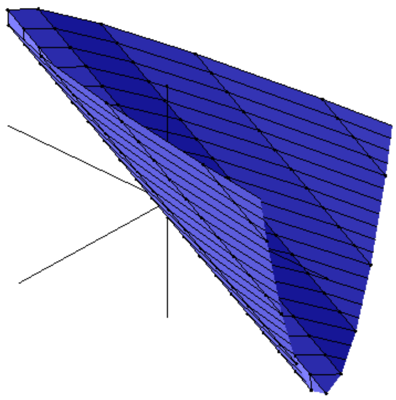
$$z = 1 \Longrightarrow 1 = x^2 - y \Longrightarrow y = x^2 - 1$$

$$z = 2 \Longrightarrow 2 = x^2 - y \Longrightarrow y = x^2 - 2$$

These are all parabolas that are either shifted up or down. Here is what the level curves look like:



iii. **Sketch in 3D.** At height $z = -1$ sketch the level curve for $z = -1$ parallel to the xy-plane. At height $z = 0$ sketch the level curve for $z = 0$ on the xy-plane. At height $z = 1$ sketch the level curve for $z = 1$ parallel to the xy-plane. As so forth to get:



(d) *Graphing and Surface Curves*: A function of the form $T = f(x, y, z)$ has 4 dimensions and thus cannot be graphed in the conventional sense. However, we can draw the traces of such a graph since they will be in 3 dimensions. When we fix different values of T and graph the resulting 3D surfaces, we call these surface curves. You have one such problem in your homework (problem 60). For this problem fix $T = 0$, $T = 1$, $T = 2$ and describe in words the 3D shapes that you get. You may need to look back at 12.6.