13.1 Intro to Vector Curves

Notation: We will write two- and three-dimensional parametric curves in either of the equivalent forms

$$x = f(t), y = g(t), z = h(t)$$

or the vector form

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle .$$

Note: The vector form gives a *position* vector that points from the origin to the curve.

Visualization

We can try to visualize the path of the motion using the following tools:

1. Eliminate the parameter to get equations relating x, y, and z. Then visualize/name the resulting surface(s) over which the motion is occurring.
2. Plot points by choosing values of t and plotting (x, y, z). (Usually, $t = 0$, $t = 1$, and $t = 2$, but look at your functions and use reasonable inputs). Connect the dots and put arrows on the curve in the direction of increasing t.
3. Use the calculus tools and measurements that we will be discussing in 13.2, 13.3 and 13.4 to learn more.

Note on Circular/Elliptical Motion: We will often do examples that involve circular motion. Remember from precalculus and calculus 1 that $x = r\cos(t)$, $y = r\sin(t)$ describe circular motion with radius r. In other words:

$$x = r\cos(t), y = r\sin(t) \quad \text{corresponds to motion on the curve} \quad x^2 + y^2 = r^2.$$

One way to derive/remember this is to use the (most important) trig identity $\sin^2(t)+\cos^2(t) = 1$. If you know $x = r\cos(t)$ and $y = r\sin(t)$, then $\cos(t) = x/r$ and $\sin(t) = y/r$ and you can eliminate the parameter by noting $\cos^2(t) + \sin^2(t) = 1$ implies $(x/r)^2+(y/r)^2 = 1$ which simplifies to $x^2+y^2 = r^2$. The same idea works for elliptical motion (see example below).

Example: Consider the motion $x = 4\cos(t)$, $y = 5\sin(t)$, $z = t$. Eliminate the parameter and describe the motion.

1. Since $z = t$, we have $x = 4\cos(z)$ and $y = 5\sin(z)$. So the motion is on the intersection of these two surfaces (which doesn't help me visualize much). But, we can go further by noting that $\cos(z) = x/4$ and $\sin(z) = y/5$ and, since $(\cos(z))^2 + (\sin(z))^2 = 1$, we have $(x/4)^2 + (y/5)^2 = 1$. Therefore all the motion occurs must satisfay $x^2/16 + y^2/25 = 1$. This is an Elliptic Cylinder in 3D. Draw it!
2. Now we can plot points.
   - $t = 0$ corresponds to $(x, y, z) = (4, 0, 0)$,
   - $t = \pi/2$ corresponds to $(x, y, z) = (0, 5, \pi/2)$.
   - $t = \pi$ corresponds to $(x, y, z) = (-4, 0, \pi)$.

   Thus, this curve is spiraling upward around an Elliptic Cylinder.
   See the 13.1 lecture notes and the textbook for several more examples.

See the next page for notes on intersections of curves and surfaces.

Intersections Notes
For all intersection questions, you are trying to combine the given conditions.

1. *Finding the intersection of two curves*: Use DIFFERENT parameters, then find if there are any set of parameters that make the x's, y's, and z's equal (make them equal and solve). If it is possible to make x, y and z equal, then the curves intersect.

   Example: Find all intersection points of $x = t^2$, $y = 4 - t$, $z = 5$ and $x = 4$, $y = u + 3$, $z = 8 - u$.
   Answer:
   In order for the x-coordinates to be equal, we need $t^2 = 4$ which implies $t = \pm 2$. So there are two cases to consider. Now we look at the y-coordinates:
   If $t = 2$, then $y = 2$ which implies $2 = u + 3$, so $u = -1$. But that does NOT work to make the z-coordinates equal.
   If $t = -2$, then $y = 6$ which implies $6 = u + 3$, so $u = 3$. That DOES make the z-coordinates equal.
   Thus, the curves intersect when $t = -2$ and $u = 3$ and the intersection point is (4, 6, 5).

2. *Finding if two moving objects collide*: Do the same thing as above. If the two time parameters are equal at an intersection, then the objects collide. If the two time parameters are not equal, then they don't collide (they are not at the intersection at the same time).

   Example: Two objects start moving at the same time. Let t be time. One object moves according to $x = t^2$, $y = 4 - t$, $z = 5$ and another moves according to $x = 4$, $y = t + 3$, $z = 8 - t$. Do they ever collide?
   Answer: NO. This is the same example from above. We found the curves only intersect when the first parameter is -2 and the second parameter is 3 (which are NOT the same time).

3. *Finding a parameterization for a curve of intersection for two surface*: There are many correct answers to a question like this. Here are a few quick methods:

   - Pick $x = t$ and solve for z and y in terms of t,
     or Pick $y = t$ and solve for x and z in terms of t,
     or Pick $z = t$ and solve for x and y in terms of t.
   - If there is a circle or ellipse, use $\cos(t)$ and $\sin(t)$.

   Examples:

   (a) Find a parameterization of the curve of intersection of $z = 3x + y^2$ and $y = 5x^2$.
   Notice everything can be written in terms of x.
   One answer:
   Let $x = t$, then $y = 5t^2$ and $z = 3t + 25t^4$. That is one parameterization of the curve of intersection.
   Another answer:
   Let $x = 13t$, then $y = 5(13t)^2$ and $z = 3(13t) + 25(13t)^4$. That is another parameterization of the curve of intersection. (I'm being silly and just showing you we could have picked some other function of t to start with)

   (b) Find a parameterization of the curve of intersection of $z = xy^2$ and $x = 2y^3$.
   Notice everything can be written in terms of y.
   One answer:
   Let $y = t$, then $x = 2t^3$ and $z = (2t^3)t^2 = 2t^5$. That is one parameterization of the curve of intersection.
   Another answer:
   Let $y = t^5 - 7$, then $x = 2(t^5 - 7)^3$ and $z = (2(t^5 - 7)^3)(t^5 - 7)^2$. That is another parameterization of the curve of intersection. (Again, I'm just showing you we could have picked some other function of t to start with, the first answer is much cleaner)

   (c) Find a parameterization of the curve of intersection of $x^2 + y^2 = 9$ and $z = 2x + y$.
   Notice that x and y are restricted to a circle!
   One answer: $x = 3\cos(t)$, $y = 3\sin(t)$, and $z = 2(3\cos(t)) + (3\sin(t)) = 6\cos(t) + 3\sin(t)$. That is one parameterization.