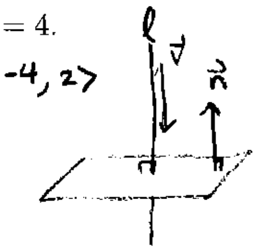
1. (13 pts)

(a) Multiple choice (circle **all** that apply, no work needed):

i. Let ℓ be the line $\mathbf{r}(t) = \langle -3t, 2t+7, 6-t \rangle$ and $\mathcal{P}$ the plane $6x - 4y + 2z = 4$. Then ℓ is...

$\vec{v} = \langle -3, 2, -1 \rangle$ IS PARALLEL TO THE NORMAL $\langle 6, -4, 2 \rangle$



(i) parallel to $\mathcal{P}$ (ii) orthogonal to $\mathcal{P}$

(iii) contained in $\mathcal{P}$ (iv) intersects $\mathcal{P}$ at one point

ii. Which of the following vector functions give points that are always on the curve of intersection of $x^2 + z^2 = 4$ and $x = 2y$:

$x \overset{\checkmark}{=} 2y$ $x^2 + z^2 \overset{\checkmark}{=} 4$ $x \overset{\checkmark}{=} 2y$, $x^2 + z^2 \overset{\checkmark}{=} 4$

(i) $\left\langle t, \frac{1}{2}t, \sqrt{4-t^2} \right\rangle$ (ii) $\langle 2\cos(t), \cos(t), 2\sin(t) \rangle$

(iii) $\langle 2\sin(t^3), \sin(t^3), 2\cos(t^3) \rangle$ (iv) $\langle 2t, t, 0 \rangle$

$x \overset{\checkmark}{=} 2y$ $x^2 + z^2 \overset{\checkmark}{=} 4$ $x \overset{\checkmark}{=} 2y$ $x^2 + z^2 = 4t^2 \neq 4$

(b) A line, L, passes thru $(1, 2, 2)$ and the *center* of the sphere, S, given by $x^2 + y^2 + z^2 - 6z = 27$. Find all (x, y, z) points of intersection of the line, L, and the sphere, S.

$$x^2 + y^2 + z^2 - 6z + 9 = 27 + 9$$

$$x^2 + y^2 + (z-3)^2 = 36 \Rightarrow \text{CENTER} = (0, 0, 3)$$

LINE: $x = 0 + (1-0)t,\ y = 0 + (2-0)t,\ z = 3 + (2-3)t$

$x = t,\ y = 2t,\ z = 3 - t$

INTERSECTION:

$$t^2 + (2t)^2 + (3-t-3)^2 \overset{?}{=} 36$$

$$t^2 + 4t^2 + t^2 = 36 \Rightarrow t^2 = 6 \Rightarrow t = \pm\sqrt{6}$$

POINTS:

$$\boxed{(x, y, z) = (-\sqrt{6}, -2\sqrt{6}, 3 + \sqrt{6}) \text{ AND } (\sqrt{6}, 2\sqrt{6}, 3 - \sqrt{6})}$$

2

2. (12 pts)

(a) Find an equation for the surface consisting of all points that are equidistant from the x-axis and the point $(0,0,1)$ **AND** give a precise name for the corresponding 3D surface.

"DIST. FROM (x,y,z) TO $\underbrace{\text{X-AXIS}}_{(x,0,0)}$" $\stackrel{?}{=}$ "DIST FROM (x,y,z) TO $(0,0,1)$"

$$\boxed{\sqrt{(x-x)^2+(y-0)^2+(z-0)^2} \stackrel{?}{=} \sqrt{(x-0)^2+(y-0)^2+(z-1)^2}}$$

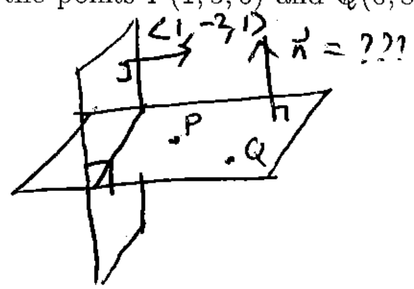
$$\Rightarrow \quad \cancel{y^2}+z^2 \stackrel{?}{=} x^2+\cancel{y^2}+(z-1)^2$$

$$\cancel{z^2} = x^2+\cancel{z^2}-2z+1$$

$$\boxed{2z = x^2+1}$$

SURFACE NAME: $\boxed{\text{PARABOLIC CYLINDER}}$

(b) Find the equation of the plane that contains the points $P(1,5,0)$ and $Q(0,8,2)$ and is orthogonal to the plane $x-2y+z=10$.



SINCE $x-2y+z=10$ IS ORTHOGONAL TO THE DESIRED PLANE, $\langle 1,-2,1\rangle$ MUST BE PARALLEL TO THE DESIRED PLANE.

IN ADDITION, $\overrightarrow{PQ} = \langle -1,3,2\rangle$ IS ALSO PARALLEL TO THE DESIRED PLANE.

THUS,
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -1 & 3 & 2 \end{vmatrix} = (-4-3)\vec{i} - (2-{-1})\vec{j} + (3-2)\vec{k}$$
$$= \langle -7,-3,1\rangle \text{ IS ORTHOGONAL TO THE DESIRED PLANE.}$$

$$\boxed{-7(x-1)-3(y-5)+z=0} \Rightarrow -7x+7-3y+15+z=0$$
$$\Rightarrow -7x-3y+z=-22$$

OR

$$-7(x-0)-3(y-8)+(z-2)=0$$ SAME

OR ANY MULTIPLE (NON-ZERO)

3. (13 pts)

(a) For some curve, $\mathbf{r}(t)$, you are given $\mathbf{T}(0) = \left\langle \frac{3}{5}, 0, -\frac{4}{5} \right\rangle$ = 'unit tangent at $t = 0$'.
(The three parts below are short answer)

i. If $|\mathbf{r}'(0)| = 3$, then give the vector $\mathbf{r}'(0) = 3\vec{T}(0) = \boxed{\left\langle \frac{9}{5}, 0, \frac{-12}{5} \right\rangle}$

ii. If you are told that $\mathbf{N}(0) = \langle 0, a, b\rangle$ = 'the principal unit normal at $t = 0$' for some numbers a and b, then what can you conclude about a and b? (list all possibilities)

$\vec{T}(0) \cdot \vec{N}(0) = 0$

$\Rightarrow 0 + 0 - \frac{4}{5}b = 0$

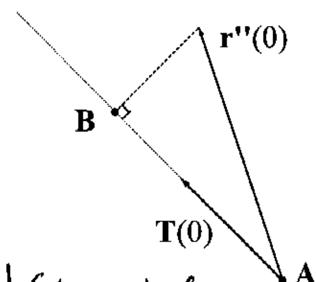
$a = -1$ or $+1$

$b = 0$

$\langle 0, a, 0\rangle$ MUST BE A UNIT VECTOR

SO $a = -1$ or $+1$

iii. If $\mathbf{r}''(0) = \left\langle \frac{5}{4}, 2, -\frac{5}{3} \right\rangle$ is drawn tail-to-tail with $\mathbf{T}(0)$ as shown below. What is the distance from point A to point B in the picture?



$$\text{COMP}_{\vec{T}(0)}\, \vec{r}\,''(0) = \frac{\frac{5}{4}\cdot\frac{3}{5} + 0 + \left(-\frac{5}{3}\right)\left(\frac{-4}{5}\right)}{|\vec{T}|}$$

$$= 3/4 + 4/3 = \frac{9+16}{12} = \boxed{\frac{25}{12}}$$

NOTE: THIS IS THE SAME AS a_T = tangential component of acceleration.

(b) Find $\mathbf{p}(t)$ if $\mathbf{p}'(t) = \langle 3t^2, 2te^{t^2}, 4te^t\rangle$ and $\mathbf{p}(0) = \langle 0, 0, 0\rangle$.

$$\int \langle 3t^2, 2te^{t^2}, 4te^t\rangle dt$$

$\boxed{\text{I}}$ $x = t^3 + c_1 \Rightarrow 0 = 0^3 + c_1 \Rightarrow c_1 = 0$

$\boxed{\text{II}}$ $\int 2te^{t^2} dt$ $\quad u = t^2$, $du = 2t\,dt$

$\int e^u du = e^u + c_2 = e^{t^2} + c_2 \Rightarrow 0 = e^0 + c_2 \Rightarrow c_2 = -1$

$\boxed{\text{III}}$ $\int 4te^t dt$ $\quad u = 4t$, $dv = e^t dt$, $du = 4dt$, $v = e^t$

$= 4te^t - \int 4e^t dt$

$= 4te^t - 4e^t + c_3 \Rightarrow 0 = 0 - 4 + c_3 \Rightarrow c_3 = 4$

$$\boxed{\left\langle t^3, e^{t^2} - 1, 4te^t - 4e^t + 4 \right\rangle}$$

4. (12 pts)

(a) Find the angle of intersection (to the nearest degree) of the two curves

$$\mathbf{r}_1(t) = \langle t, 6-2t, 15+t^2 \rangle \text{ and } \mathbf{r}_2(s) = \langle 5-s, 2s-4, s^2 \rangle.$$

$$\left.\begin{array}{l} t \overset{?}{=} 5-s \\ 6-2t \overset{?}{=} 2s-4 \end{array}\right\} \Rightarrow 6-2(5-s)=2s-4 \Rightarrow 6-10+2s=2s-4 \Rightarrow 0=0$$

MOVE ON TO THIRD EQUATION

$$15+t^2 \overset{?}{=} s^2 \qquad 15+(5-s)^2 = s^2 \Rightarrow 15+25-10s+\cancel{s^2} = \cancel{s^2}$$

$$\Rightarrow 40 = 10s \Rightarrow \boxed{s=4} \Rightarrow \boxed{t=1}$$

$$\vec{r}_1{}'(t) = \langle 1,-2,2t \rangle \qquad \vec{r}_2{}'(s) = \langle -1,2,2s \rangle$$

$$\vec{r}_1{}'(1) = \langle 1,+2,2 \rangle \qquad \vec{r}_2{}'(4) = \langle +1,-2,8 \rangle$$

$$\cos(\theta) = \frac{-1+4+16}{\sqrt{1+4+4}\,\sqrt{1+4+64}} = \frac{11}{3\sqrt{69}} \Rightarrow \theta = \boxed{\cos^{-1}\left(\frac{11}{3\sqrt{69}}\right) \approx 64^\circ}$$

(b) For $x > 0$, find the x-value at which curvature is maximum for $f(x) = \dfrac{x^3}{3}$.

$$f'(x) = x^2 \quad f''(x) = 2x$$

$$\kappa(x) = \frac{|2x|}{(1+(x^2)^2)^{3/2}} = \frac{2x}{(1+x^4)^{3/2}}$$

$$\kappa'(x) = \frac{(1+x^4)^{3/2}\cdot 2 - \cancel{2}x\,\frac{3}{\cancel{2}}\cdot 4x^3(1+x^4)^{1/2}}{(1+x^4)^3} \overset{?}{=} 0$$

$$\Rightarrow 2(1+x^4)^{1/2}\left[1+x^4-6x^4\right] \overset{?}{=} 0$$

$$\Rightarrow 1-5x^4 \overset{?}{=} 0$$

$$x^4 = 1/5$$

$$\boxed{x = (1/5)^{1/4}}$$