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Monday, April 17, 2017 8:33 AM

1. (11 points)

(a) (5 pts) Consider the line through the points $P(1,3,-2)$ and $Q(3,5,7)$. Find the (x,y,z) coordinates of the point at which this line intersects the xz-plane.

DIRECTION: $\vec{PQ} = \langle 2,2,9 \rangle$

LINE: $x = 1+2t$
$y = 3+2t$
$z = -2+9t$

ASIDE: There are infinitely many parametrizations of the line. But the direction must be parallel to $\langle 2,2,9\rangle$

INTERSECT xz-PLANE: $y=0 \Rightarrow 0 = 3+2t \Rightarrow t = -3/2$

$$\boxed{\begin{aligned}(x,y,z) &= \left(1+2\left(-\tfrac{3}{2}\right), 0, -2+9\left(-\tfrac{3}{2}\right)\right)\\ &= \left(-2, 0, -\tfrac{31}{2}\right)\end{aligned}}$$

← Everyone should get the same answer here

(b) Consider the **plane**, P, **that contains the point** $(1,-1,2)$ and is the orthogonal to the line given by

$$L: \begin{cases} x = -3t \\ y = 2+7t \\ z = 5-t \end{cases}$$

i. (4 pts) Find the equation for the plane, P.

$\langle -3,7,-1\rangle$ is normal to the desired plane.

$\langle 1,-1,2\rangle$ is a position vector.

So
$$\boxed{\begin{aligned}&\langle -3,7,-1\rangle \cdot \langle x-1, y+1, z-2\rangle = 0\\ &-3(x-1) + 7(y+1) - (z-2) = 0\\ &-3x+3+7y+7-z+2=0\\ &-3x+7y-z+12=0\end{aligned}}$$

ii. (2 pts) At what point (x,y,z) does this plane intersect the x-axis?

x-axis $\Leftrightarrow$ $y=0$ and $z=0$

So $-3x+7(0)-(0)+12=0 \Rightarrow x=4$

$$\boxed{(4,0,0)}$$

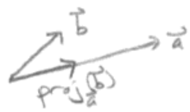
2. (14 points)

(a) (6 pts) Assume **a** and **b** are nonzero three-dimensional vectors that are not parallel and are not perpendicular.

In each case below, determine if the two vectors are *always are orthogonal*, *always are parallel*, *always are neither parallel or perpendicular*, or *it depends on the vectors* (meaning depending on the vectors it is possible they could be perpendicular or parallel or neither).

Circle one for each (no work is necessary):

i. $\underbrace{\mathbf{a}\times\mathbf{b}}_{\text{orthogonal to both } \vec{a} \text{ and } \vec{b}}$ and $2\mathbf{b}$. ← same direction as $\vec{b}$	orthogonal	parallel	neither	depends
ii. $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$	orthogonal	parallel	neither	depends
iii. $\mathbf{proj_a}(\mathbf{b})$ and $\mathbf{b}$.	orthogonal	parallel	neither	depends
iv. $\mathbf{proj_a}(\mathbf{b})$ and $\frac{1}{\lvert\mathbf{a}\rvert}\mathbf{a}$.	orthogonal	parallel	neither	depends
v. $\mathbf{a}-\mathbf{b}$ and $\mathbf{b}-\mathbf{a}$.	orthogonal	parallel	neither	depends



(b) (8 pts) Consider the three points $A(1,3,4)$, $B(0,2,1)$, $C(2,3,6)$.

i. Find the area of the triangle determined by the three points.

$\overrightarrow{AB} = \langle -1,-1,-3\rangle$, $\overrightarrow{AC} = \langle 1,0,2\rangle$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & -3 \\ 1 & 0 & 2 \end{vmatrix} = (-2-0)\vec{i} - (-2-(-3))\vec{j} + (0-(-1))\vec{k} = \langle -2,-1,1\rangle$$

$$\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\sqrt{4+1+1} = \boxed{\frac{1}{2}\sqrt{6}}$$

ii. For this same triangle, find the angle at the corner B.
(Give in degrees rounded to two places after the decimal).

ASIDE: IN RADIANS THIS IS 0.135219 rad

$\overrightarrow{BA} = \langle 1,1,3\rangle$, $\overrightarrow{BC} = \langle 2,1,5\rangle$

$$\langle 1,1,3\rangle\cdot\langle 2,1,5\rangle = \sqrt{1+1+9}\,\sqrt{4+1+25}\,\cos(\theta)$$

$$2+1+15 = \sqrt{11}\,\sqrt{30}\,\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{18}{\sqrt{11}\sqrt{30}}\right) \simeq 7.74936637 6$$

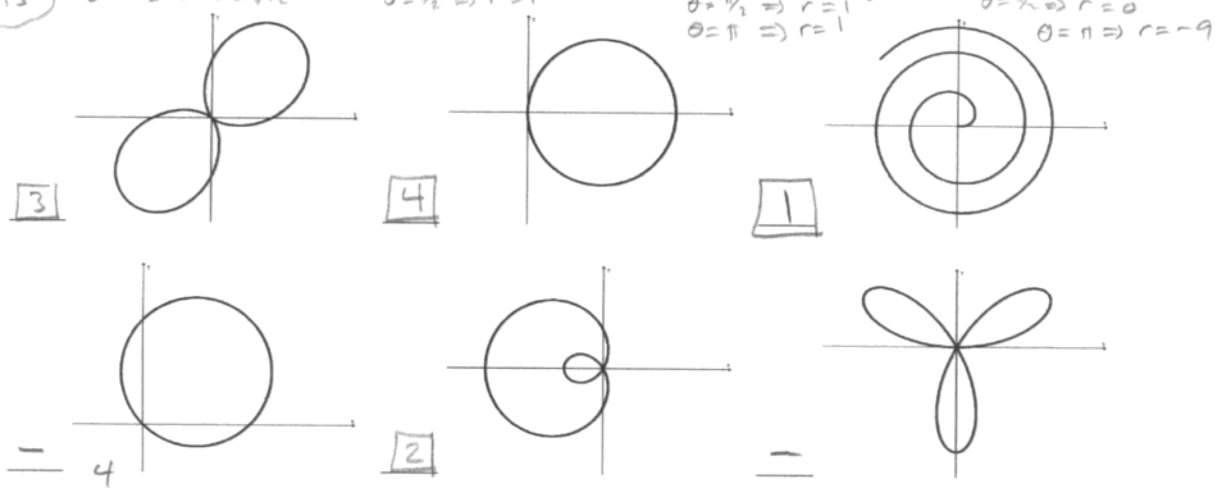
$$\simeq \boxed{7.75^\circ}$$

3. (a) (6 pts) In the blanks provided to the left of each graph, put the number of the polar equation that matches the graph in the xy-plane (two graphs will not be labeled).

1. $r = \sqrt{\theta}$ 2. $r = 1 - 2\cos(\theta)$ 3. $r = 1 + \sin(2\theta)$ 4. $r = 9\cos(\theta)$

PLOT POINTS

1. $\theta = 0 \Rightarrow r = 0$ ✓; $\theta = \pi/2 \Rightarrow r = \sqrt{\pi/2}$
2. $\theta = 0 \Rightarrow r = -1$ ✓; $\theta = \pi/2 \Rightarrow r = 1$
3. $\theta = 0 \Rightarrow r = 1$ ✓; $\theta = \pi/2 \Rightarrow r = 1$; $\theta = \pi \Rightarrow r = 1$
4. $\theta = 0 \Rightarrow r = 9$ ✓; $\theta = \pi/2 \Rightarrow r = 0$; $\theta = \pi \Rightarrow r = -9$



(b) (3 pts) Find the (x, y) coordinates of all points on the curve $r = 1 + \sin(2\theta)$ that intersect the line $y = x$.

INTERSECT $y = x \Rightarrow \theta = \pi/4$ or $\theta = 5\pi/4$ or $r = 0$

ALSO SEE PICTURE

$$r = 1 + \sin(2\pi/4) = 2 \qquad r = 1 + \sin(2 \cdot 5\pi/4), \quad r = 2$$

$$(r, \theta) = (2, \pi/4) \Rightarrow x = 2\cos(\pi/4) = 2\sqrt{2}/2, \quad y = 2\sin(\pi/4) = 2\sqrt{2}/2$$

or by symmetry

$$(r, \theta) = (2, 5\pi/4) \Rightarrow x = 2\cos(5\pi/4) = -2\sqrt{2}/2, \quad y = 2\sin(5\pi/4) = -2\sqrt{2}/2$$

$$r = 0 \Rightarrow x = 0, \quad y = 0$$

$$\boxed{(\sqrt{2}, \sqrt{2}), \ (-\sqrt{2}, -\sqrt{2}), \ (0, 0)}$$

4. (a) (10 pts) Consider the vector function $\mathbf{r}(t) = \langle t^2 - 2t, t^3 - 4t\rangle$ and the corresponding parametric curve $x = t^2 - 2t$, $y = t^3 - 4t$.

i. Find the value of $\dfrac{d^2y}{dx^2}$ at $t = -1$.

$$\frac{dy}{dx} = \frac{3t^2-4}{2t-2} \qquad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{3t^2-4}{2t-2}\right)}{dx/dt} = \frac{\frac{(2t-2)6t - 2(3t^2-4)}{(2t-2)^2}}{2t-2}$$

$$\frac{d^2y}{dx^2} = \frac{(2t-2)6t - 2(3t^2-4)}{(2t-2)^3} \quad \text{at } t=-1$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{(-4)(-6)-2(-1)}{(-4)^3} = \frac{24+2}{-64} = -\frac{26}{64} = -\frac{13}{32}}$$

ii. Find values of t at which the tangent line is parallel to the vector $\langle 1, 2\rangle$.

TWO WAYS TO DO THIS ① SLOPE = 2 OR ② $\langle \frac{dx}{dt}, \frac{dy}{dt}\rangle = k\langle 1,2\rangle$

BOTH LEAD TO $3t^2-4 = 2(2t-2) \Rightarrow 3t^2-4 = 4t-4$

$$3t^2 = 4t$$
$$3t^2 - 4t = 0$$
$$t(3t-4) = 0$$

$\boxed{t=0}$ or $\boxed{t = 4/3}$

(b) (5 pts) Find parametric equations for the tangent line to the curve given by $\mathbf{r}(t) = \langle 2\sin(3t), 3t, -2t\cos(t)\rangle$ at the time $t = \frac{\pi}{3}$.
(Give exact, simplified, numbers in your answer).

$$\vec{r}(\pi/3) = \langle 0, \pi, -\pi/3\rangle$$

$$\vec{r}\,'(t) = \langle 6\cos(3t), 3, -2\cos(t) + 2t\sin(t)\rangle$$
$$\vec{r}\,'(\pi/3) = \langle -6, 3, -1 + 2\tfrac{\pi}{3}\tfrac{\sqrt{3}}{2}\rangle$$

$$\boxed{\begin{aligned} x &= 0 - 6t \\ y &= \pi + 3t \\ z &= -\pi/3 + \left(-1 + \tfrac{\pi\sqrt{3}}{3}\right)t \end{aligned}}$$