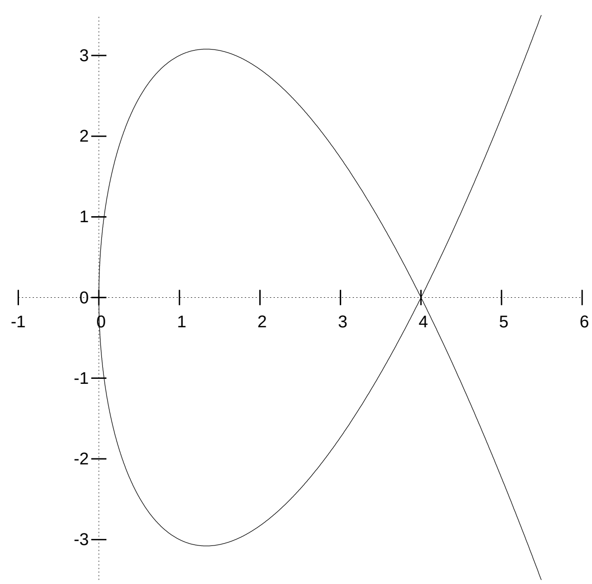
Math 126 C - Spring 2009
Mid-Term Exam Number Two
May 14, 2009
Solutions

1. *The curve $y = x^x$ has one local extremum. Find the curvature at that point.*

   We have
$$y = x^x = e^{x \ln x}$$
   so
$$y' = e^{x \ln x}(\ln x + 1)$$
   and
$$y'' = e^{x \ln x}((\ln x + 1)^2 + \frac{1}{x}).$$
   At a local extremum, $y' = 0$. Solving $y' = 0$, we find
$$x = \frac{1}{e}$$
   At this point we have curvature
$$\kappa = \frac{|y''|}{(1 + (y')^2)^{3/2}} = |y''| = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left(\frac{1}{\frac{1}{e}}\right) = e\left(\frac{1}{e}\right)^{1/e} = e^{1-\frac{1}{e}}.$$

2. *An object moves so that its position at time t is given by*
$$\vec{r}(t) = \langle t^2, t^3 - 4t, 0 \rangle.$$
   A portion of its path is shown below.



Find all times when the object's velocity vector is orthogonal to its acceleration vector.

We find
$$\vec{r}'(t) = \langle 2t, 3t^2 - 4, 0\rangle$$
and
$$\vec{r}''(t) = \langle 2, 6t, 0\rangle$$
so that
$$\vec{r}'\vec{r}'' = 4t + 6t(3t^2 - 4) = t(-20 + 18t^2) = 0$$
when $t = 0$ or
$$t = \pm\frac{\sqrt{10}}{3}.$$

3. *Let*
$$z = xe^y + y\sin x + \frac{x}{y}.$$

   (a) *Find* $\dfrac{\partial z}{\partial x}$.
$$\frac{\partial z}{\partial x} = e^y + y\cos x + \frac{1}{y}.$$

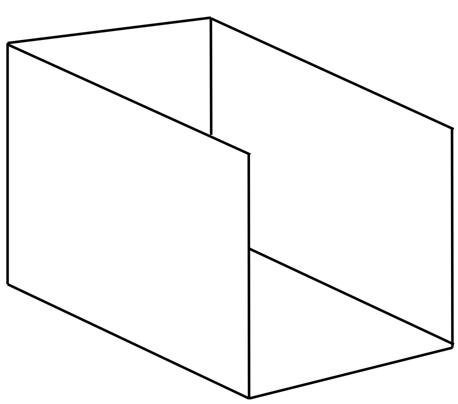
   (b) *Find* $\dfrac{\partial z}{\partial y}$.
$$\frac{\partial z}{\partial y} = xe^y + \sin x - \frac{x}{y^2}.$$

4. *You wish to build a four-sided box like the one shown in the figure, with three rectangular sides perpendicular to a rectangular base.*

   You want the box to have a volume of 100 cubic centimeters.

   If all sides are to be made from the same thin material, what dimensions will minimize the amount of material used?

   Be sure to justify your answer using the Hessian (i.e., the second derivatives test).



If we label the height of the box z, and label the sides of the base y and x, then the volume of the box is

$$xyz = 100$$

and the amount of material needed is

$$M = xy + 2yz + xz$$

Since $xyz = 100$, we have

$$z = \frac{100}{xy}$$

so we may write

$$M = M(x, y) = xy + \frac{200}{x} + \frac{100}{y}.$$

Taking partial derivatives, we have

$$\frac{\partial M}{\partial x} = y - \frac{200}{x^2}$$

and

$$\frac{\partial M}{\partial y} = x - \frac{100}{y^2}.$$

Setting these equal to zero, we can conclude first that

$$y = \frac{200}{x^2}$$

and then that

$$x^3 = 400$$

or $x \approx 7.36806...$, from which we find

$$y = z = 50^{1/3} \approx 3.684....$$

Thus, we have one critical point. Taking second partials we find that

$$\frac{\partial^2 M}{\partial x^2} = \frac{400}{x^3}$$

and

$$\frac{\partial^2 M}{\partial y^2} = \frac{200}{y^3}$$

and so

$$D = \frac{400 \cdot 200}{x^3 y^3} - 1 = 3 > 0$$

at the critical point, so we have a local min, and since it is the only critical point, we may conclude that it yields the global min.

5. *Find the volume of the solid under the surface* $z = xy^2$ *and above the triangle with vertices* $(0, 0)$, $(0, 5)$ *and* $(2, 3)$.

The required volume can be expressed as the following double integral

$$V = \int_0^2 \int_{\frac{3}{2}x}^{5-x} xy^2 \, dy \, dx$$

When evaluated, we find

$$V = \frac{82}{3}.$$