Math 126 C - Spring 2007
Mid-Term Exam Number One Solutions
April 19, 2007

1. *Let* $f(x) = e^x \sin x$.

   (a) *Find the second-order Taylor polynomial* $T_2(x)$ *for* $f(x)$ *based at* $b = 0$.

   $$T_2(x) = x + x^2$$

   (b) *Give a bound on the error* $|f(x) - T_2(x)|$ *for* x *in the interval* $-0.1 \le x \le 0.1$.
   We may use the fact that

   $$|f'''(x)| = |2e^x(\cos x - \sin x)| \le 4e^x \le 4e^{0.1}$$

   on the interval $-0.1 \le x \le 0.1$, so the error is no more than

   $$\frac{4e^{0.1}}{3!}|0.1|^3 = 0.005894.....$$

2. *Find the first four non-zero terms of the Taylor series for*

   $$f(x) = xe^{x^2} - \frac{1}{4+x^2}$$

   based at $b = 0$.
   Since

   $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

   for all x, we have

   $$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

   and

   $$xe^{x^2} = x + x^3 + \frac{x^5}{2!} + \frac{x^7}{3!} + \cdots$$

   Also,

   $$\frac{1}{4+x^2} = \left(\frac{1}{4}\right)\frac{1}{1 - \left(-\left(\frac{x}{2}\right)^2\right)} = \frac{1}{4}\sum_{k=0}^{\infty}\left(-\left(\frac{x}{2}\right)^2\right)^k = \frac{1}{4} - \frac{x^2}{4 \cdot 4} + \frac{x^4}{4 \cdot 2^4} - \frac{x^6}{4 \cdot 2^6} + \cdots$$

   Combining these results yields

   $$f(x) = -\frac{1}{4} + x + \frac{x^2}{4 \cdot 4} + x^3 + \cdots.$$

3. *Find the equation of the plane containing the line of intersection of the two planes*

$$x + y + z + 5 = 0 \text{ and } 3x + 2y - z + 2 = 0$$

and the point $(1, 2, 1)$.

We find the direction vector of the line of intersection: it is the cross product of the planes' normal vectors

$$\langle 1, 1, 1\rangle \times \langle 3, 2, -1\rangle = \langle -3, 4, -1\rangle$$

Then we need a point on the line (which is hence on the plane we seek). Choosing $x = 0$, we can find the point

$$\left(0, -\frac{7}{3}, -\frac{8}{3}\right)$$

on the line. The vector extending from this point to $(1, 2, 1)$ (the other point we know is on the plane) is

$$\langle 1, \frac{13}{3}, \frac{11}{3}\rangle$$

or, equivalently (since direction is all we need here)

$$\langle 3, 13, 11\rangle.$$

The normal vector of the plane we seek is then

$$\langle 3, 13, 11\rangle \times \langle -3, 4, -1\rangle = \langle -57, -30, 51\rangle$$

and so the equation of the plane is

$$-57x - 30y + 51z + d = 0$$

Since $(1, 2, 1)$ is on the line, we can find $d = 66$, so the plane is

$$-57x - 30y + 51z + 66 = 0.$$

4. *Find the point of intersection of the two lines*

$$x = 4 - t, y = 6 + 2t, z = -1 + 3t \text{ and } x = 1 + 2t, y = 14 - 8t, z = 7 - 4t.$$

Suppose (a, b, c) is the point of intersection. Then there is a t value, say t_1 such that

$$a = 4 - t_1, b = 6 + 2t_1, c = -1 + 3t_1$$

and a t value, say t_2 such that

$$a = 1 + 2t_2, b = 14 - 8t_2, c = 7 - 4t_2.$$

Hence, we have

$$4 - t_1 = 1 + 2t_2$$

and

$$6 + 2t_1 = 14 - 8t_2$$

This is a pair of equations in two unknowns which we can solve to get $t_1 = 2$ and $t_2 = \frac{1}{2}$. This then gives us the point of intersection

$$(4 - 2, 6 + 2 \cdot 2, -1 + 3 \cdot 2) = (2, 10, 5).$$

5. *Let S be the surface defined as the set of points p (in three-dimensional space) such that the distance from p to the plane $y = 5$ equals the distance from p to the line*

$$y = 1, z = 2.$$

(a) *Find an equation for S.*

The distance from a point (x, y, z) to the plane $y = 5$ is

$$|y - 5|$$

and the distance from (x, y, z) to the line $y = 1, z = 2$ is

$$\sqrt{(y-1)^2 + (z-2)^2}$$

so an equation of the surface (i.e., an equation that is satisfied by all points on the surface) is

$$|y - 5| = \sqrt{(y-1)^2 + (z-2)^2}.$$

After squaring the equation

$$(y - 5)^2 = (y - 1)^2 + (z - 2)^2$$

we can simplify it to

$$-8y + 24 = (z - 2)^2$$

(b) *Find the equation of the trace of S in the plane $z = 6$. Describe the trace (i.e. what kind of curve is it?).*

If $z = 6$ then the equation of the surface simplifies to

$$-8y + 24 = 4^2 = 16$$

i.e.

$$y = 1$$

In the plane $z = 6$, the set of points satisfying the equation $y = 1$ is a **line**.