Closing Thu: 12.4(1)(2), 12.5(1)
Closing next Tue: 12.5(2)(3), 12.6
Closing next Thu: 10.1/13.1
Office Hours: 1:30-3:00pm in Smith 309

12.5 Lines and Planes in 3D

Lines: We use parametric equations to describe 3D lines. Here is a 2D warm-up:

Ex: Consider the line: y = 4x + 5.

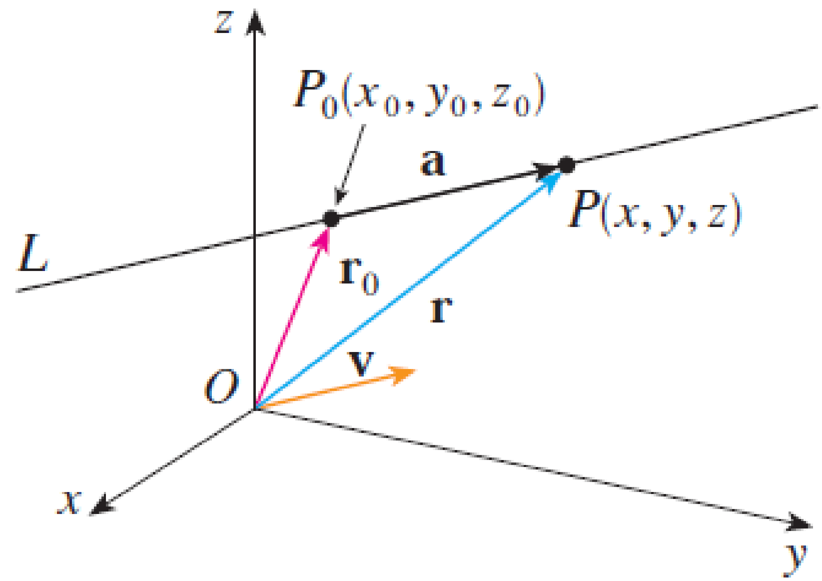
(a) Find a vector parallel to the line. Call it **v.**

(b) Find a vector whose head touches the line when drawn from the origin. Call it $\mathbf{r_0}$.

(c) Observe, we can reach all other points on the line by walking along $\mathbf{r_0}$, then adding scale multiples of **v**.

This same idea works to describe any line in 2- or 3-dimensions.

The equation for a line in 3D:
$\boldsymbol{v} = \langle a, b, c \rangle$ = parallel to the line.
$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ = a position vector
then all other points, (x, y, z), satisfy
$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \mathrm{t}\langle a, b, c \rangle$,
for some number t.
The above form ($\boldsymbol{r} = \boldsymbol{r_0} + \mathrm{t}\,\boldsymbol{v}$) is called the *vector form* of the line.



We also write this in *parametric form* as:

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct.$$

or in *symmetric form*:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Basic Example – Given Two Points:
Find parametric equations of the line thru the points P(1,0,2) and Q(-1, 2, 1).

General Line Facts

1. Two lines are **parallel** if their direction vectors are parallel.

2. Two lines **intersect** if they have an (x, y, z) point in common (use a different parameter for each line when solving!)

Note: The *acute angle of intersection* would be the acute angle between the direction vectors.

3. Two lines are **skew** if they don't intersect and aren't parallel.

<u>Planes</u>:

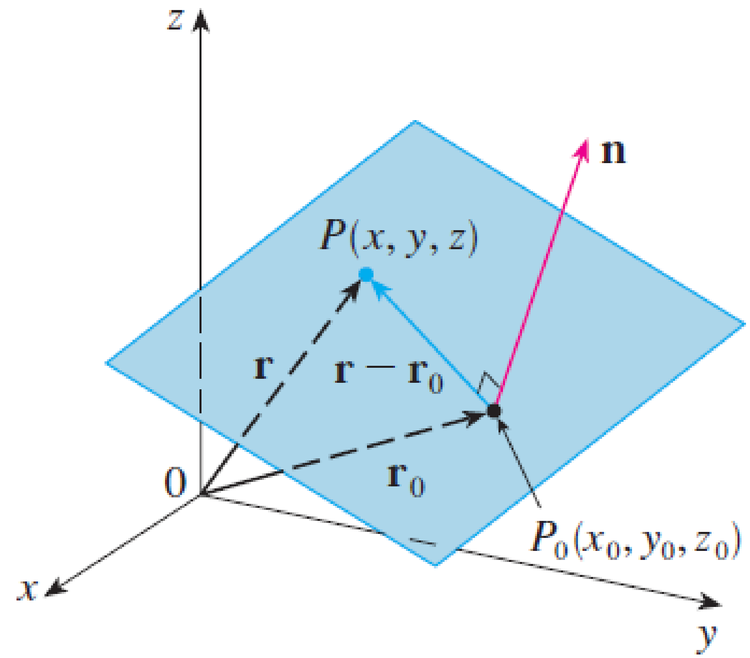
The equation for a plane in 3D:

$\boldsymbol{n} = \langle a, b, c \rangle$ = orthogonal to plane

$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ = a position vector

then all other points, (x, y, z), satisfy

$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$



The above form ($\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$) is called the *vector form* of the plane.

We also write this in *standard form* as:

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

We sometimes expand to the form

$ax + by + cz - ax_0 - by_0 - cz_0 = 0,$

letting $d = -ax_0 - by_0 - cz_0$, we get

$ax + by + cz + d = 0.$

(Note: we call this a *linear equation*)

Basic Example –

Given Three Points:

Find the equation for the plane through the points P(0, 1, 0), Q(3, 1, 4), and R(-1, 0, 0)

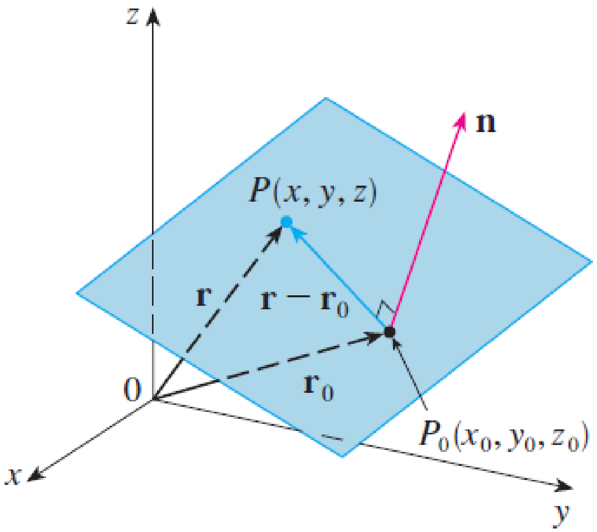
General Plane Facts

1. Two planes are **parallel** if their normal vectors are parallel.

2. If two planes are not parallel, then they must intersect to form a line.

2a. The *acute angle of intersection* is the acute angle between their normal vectors.

2b. The planes are orthogonal if their normal vectors are orthogonal.



Side comment:

If you want the distance between two *parallel* planes, then

(a) Find any point on the first plane (x_0, y_0, z_0) and any point on the second plane (x_1, y_1, z_1).

(b) Write $\mathbf{u} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

(c) Project **u** onto one of the normal vectors **n**.

$|\text{comp}_{\mathbf{n}}(\mathbf{u})|$ = dist. between planes