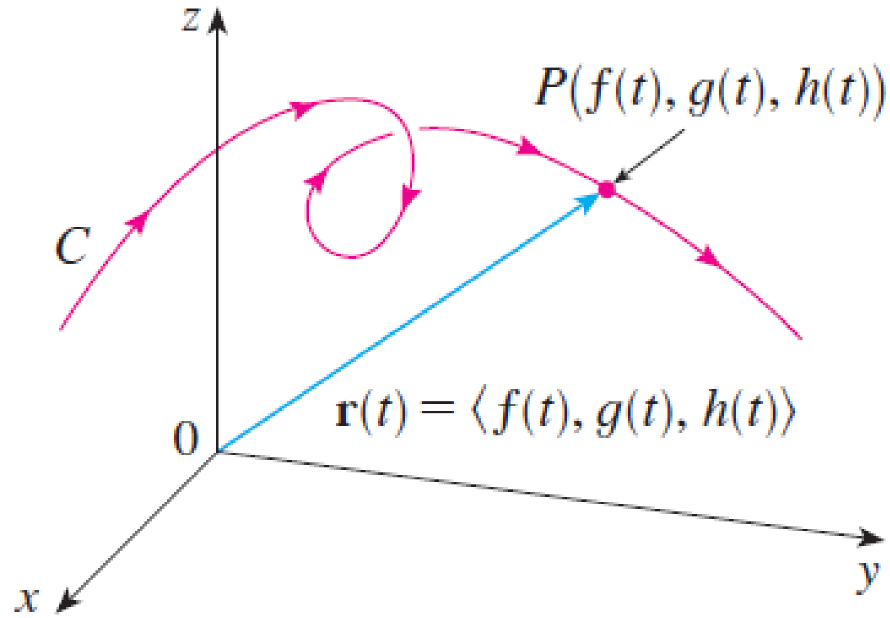
10.1/13.1 Parametric Curves Intro (2D and 3D)



Parametric equations:

x = x(t), y = y(t), z = z(t)

To plot, you select various values of *t*, compute (x(t),y(t),z(t)), and plot the corresponding (x,y,z) points.

The resulting curve is called a **parametric curve**, or **space curve** (in 3D).

We also like to write the equation in vector form:

$$\vec{\boldsymbol{r}}(t) = \langle x(t), y(t), z(t)\rangle$$

= a position vector for the curve

i.e. if the tail of this vector is drawn from the origin, the head will be at (x(t),y(t),z(t)) on the curve.

Various Parametric Facts:

1. **Eliminating the parameter**

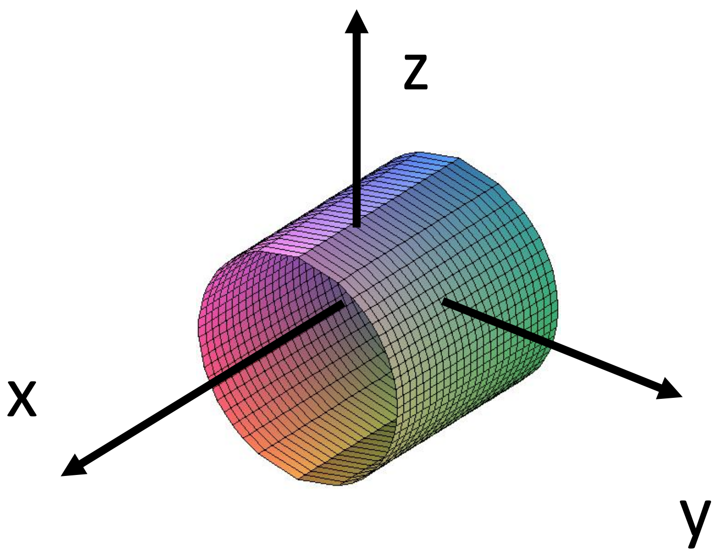
(a) Solving for *t* in one equation, substitute into the others.

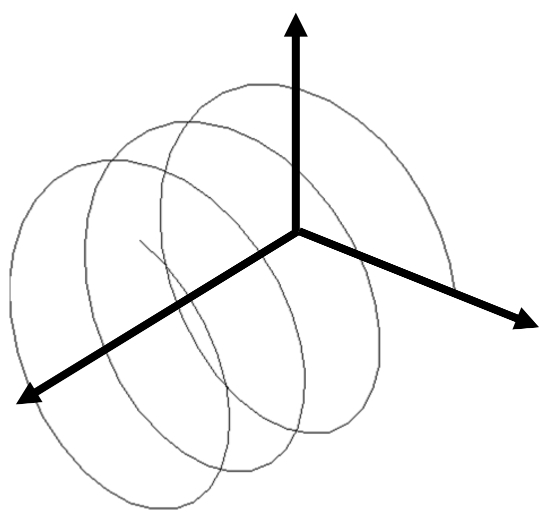
(b) Use $(\sin(u))^2 + (\cos(u))^2 = 1$.

This gives the surface/path over which the motion is occurring.

All points given by the parametric equations: $x = t$, $y = \cos(2t)$, $z = \sin(2t)$

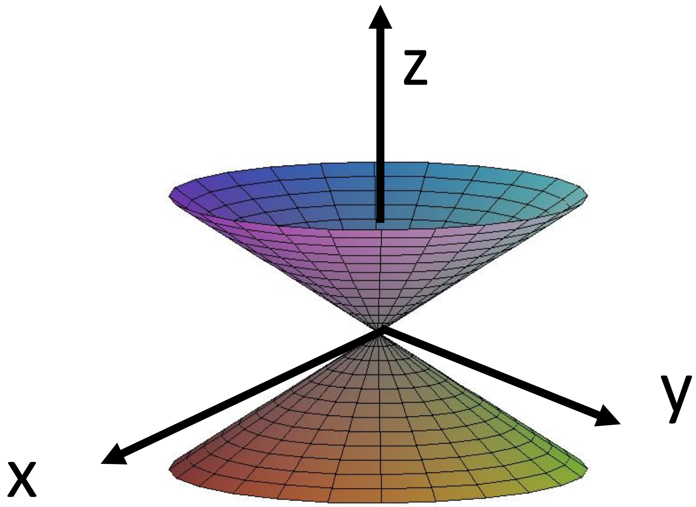
are on the cylinder: $y^2 + z^2 = 1$

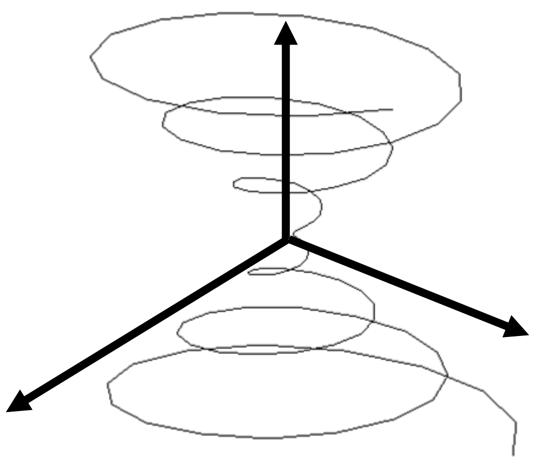




All points given by the parametric equations: $x = t\cos(t)$, $y = t\sin(t)$, $z = t$

are on the cone: $z^2 = x^2 + y^2$

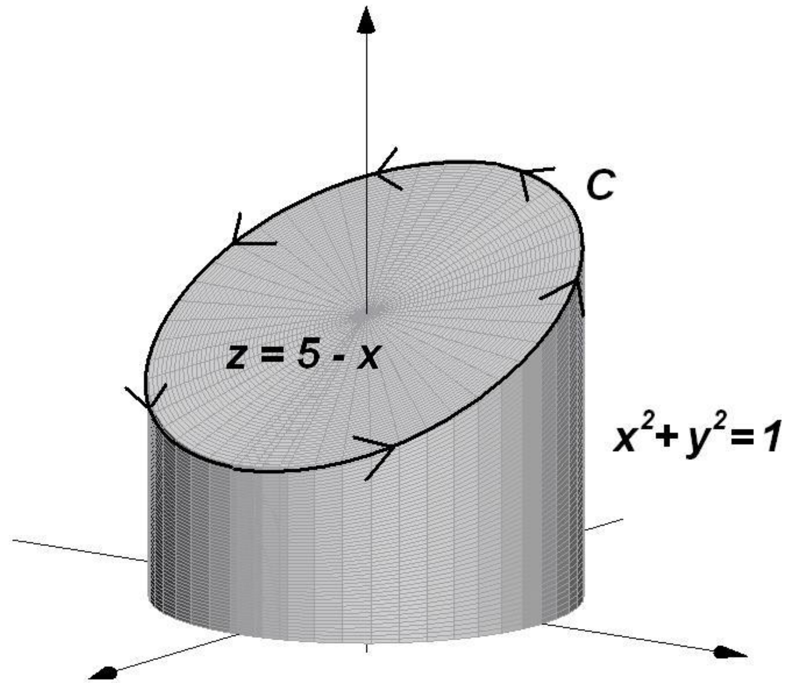




2. **Intersection issues**:

(a) *To find where two curves intersect*, use two different parameters!!!
We say the curves **collide** if the intersection happens at the same parameter value.

(b) To find parametric equations for *the intersection of two surfaces*, combine the surfaces into one equation. Let one variable be *t* and solve for the others.
(Or use sin(t), cos(t) if there is a circle involved)



3. Basic 2D Parametric Calculus

(Review of Math 124):

From the chain rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

Rearranging gives

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \text{slope.}$$

Note that if y = f(x), this says

$$\frac{d}{dx}\big(f(x)\big) = \frac{\frac{d}{dt}(f(x))}{dx/dt}.$$

So the 2nd derivative satisfies:

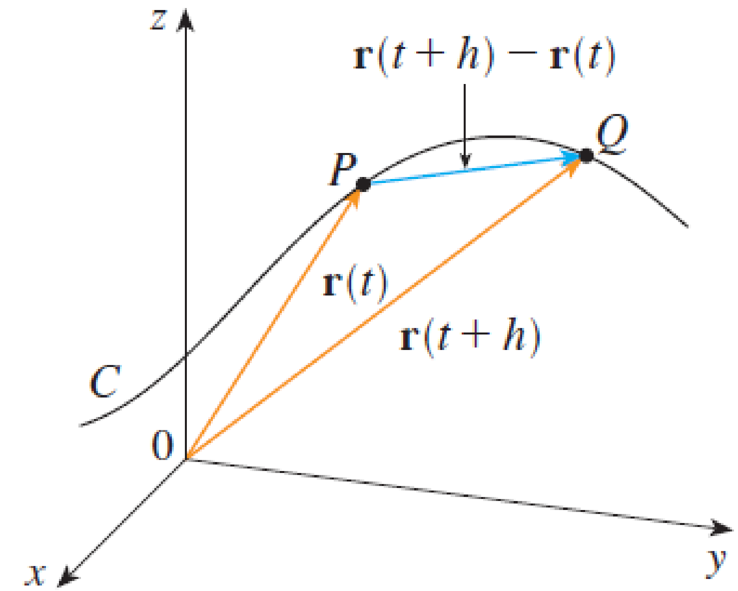
$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{\frac{d}{dt}\big(f'(x)\big)}{dx/dt}$$

4. Vector Calculus

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$,
we define

$$\vec{r}'(t) = \lim_{h\to 0} \langle \frac{x(t+h)-x(t)}{h}, \frac{y(t+h)-y(t)}{h}, \frac{z(t+h)-z(t)}{h} \rangle$$

so $\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.



We also define

$$\vec{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle.$$

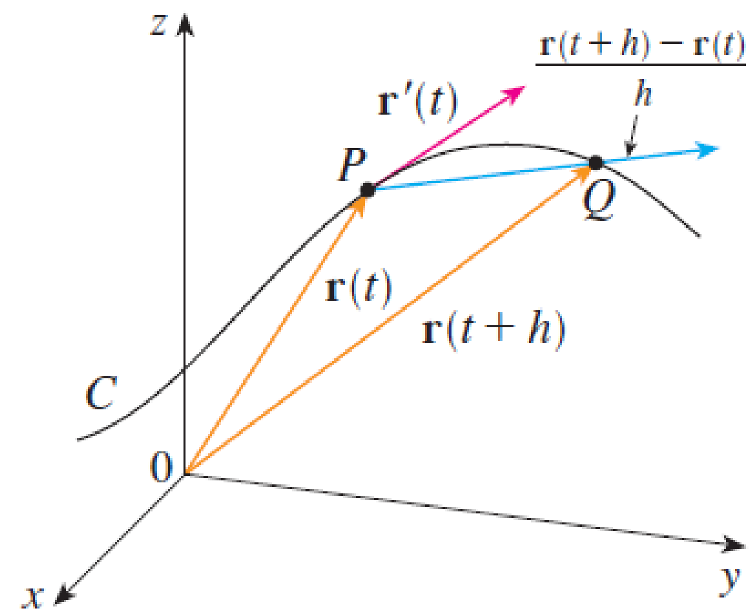
In 13.3, we will see that $\vec{r}''(t)$ gives information about the curvature.

In 13.4, we will see that

$\vec{r}'(t)$ is a velocity vector,

$|\vec{r}'(t)|$ is the speed, and

$\vec{r}''(t)$ is an acceleration vector.



We also define

$$\int \vec{r}(t)dt = \langle \int x(t)dt, \int y(t)dt, \int z(t)dt \rangle.$$

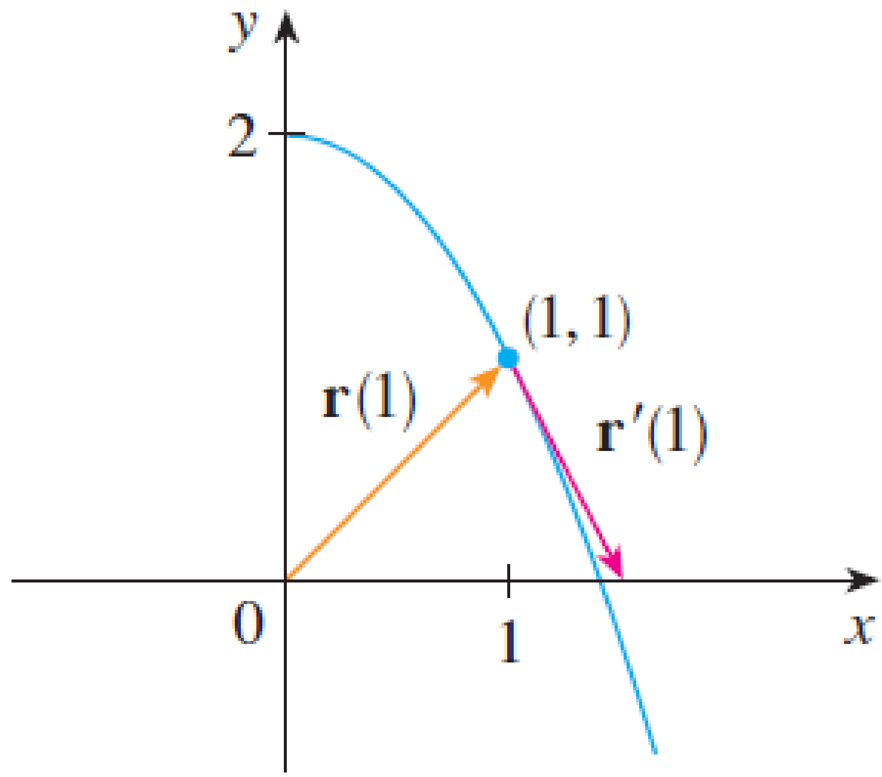
Morale, do derivatives and integral **component-wise**.

Ex: Consider $\vec{\boldsymbol{r}}(t) = \langle t, \cos(2t), \sin(2t) \rangle$.

(a) Find $\vec{\boldsymbol{r}}'(t)$, $|\vec{\boldsymbol{r}}'(t)|$, and $\vec{\boldsymbol{r}}''(t)$.

(b) Find $\vec{\boldsymbol{r}}(\pi/4)$ and $\vec{\boldsymbol{r}}'(\pi/4)$.

(c) Give the equation for the tangent line at $t = \pi/4$


y
2
(1, 1)
r(1)
r′(1)
0
1
x

5. Arc Length

The length of a curve from *t = a* to *t = b* is given by

$$\int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\, dt$$

$$= \int_a^b |\vec{\boldsymbol{r}}'(t)| dt$$

(Note: 2D is same without the z'(t)).

We call this **arc length**.

The arc length from 0 to *u* is often written as

$$s(u) = \int_0^u |\vec{\boldsymbol{r}}'(t)| dt$$

We call this the **arc length function**.