10.1/13.1 Parametric Curves Intro
(2D and 3D)

Parametric equations:
\[ x = x(t), \quad y = y(t), \quad z = z(t) \]

To plot, you select various values of \( t \), compute \((x(t), y(t), z(t))\), and plot the corresponding \((x, y, z)\) points.

The resulting curve is called a **parametric curve**, or **space curve** (in 3D).

We also like to write the equation in vector form:
\[ \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \]

i.e. if the tail of this vector is drawn from the origin, the head will be at \((x(t), y(t), z(t))\) on the curve.
Various Parametric Facts:

1. **Eliminating the parameter**
   
   (a) Solving for $t$ in one equation, substitute into the others.
   
   (b) Use $(\sin(u))^2 + (\cos(u))^2 = 1$.

This gives the surface/path over which the motion is occurring.
All points given by the parametric equations: \( x = t, y = \cos(2t), z = \sin(2t) \)

are on the cylinder: \( y^2 + z^2 = 1 \)
All points given by the parametric equations: \( x = t\cos(t), \ y = t\sin(t), \ z = t \) are on the cone: \( z^2 = x^2 + y^2 \)
2. **Intersection issues:**

(a) *To find where two curves intersect,* use two different parameters!!!

We say the curves *collide* if the intersection happens at the same parameter value.

(b) To find parametric equations for the *intersection of two surfaces,* combine the surfaces into one equation. Let one variable be \( t \) and solve for the others.

(Or use \( \sin(t), \cos(t) \) if there is a circle involved)
3. Basic 2D Parametric Calculus
(Review of Math 124):

From the chain rule \( \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \).

Rearranging gives
\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \text{slope}.
\]

Note that if \( y = f(x) \), this says
\[
\frac{d}{dx} (f(x)) = \frac{d}{dt} (f(x)) \cdot \frac{dx}{dt}.
\]

So the 2\(^{nd}\) derivative satisfies:
\[
f''(x) = \frac{d}{dx} (f'(x)) = \frac{d}{dt} (f'(x)) \cdot \frac{dx}{dt}.
\]
4. Vector Calculus

If \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \), we define
\[
\mathbf{r}'(t) = \lim_{h \to 0} \left\langle \frac{x(t + h) - x(t)}{h}, \frac{y(t + h) - y(t)}{h}, \frac{z(t + h) - z(t)}{h} \right\rangle
\]
so \( \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \).

We also define
\[
\mathbf{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle.
\]
In 13.3, we will see that \( \mathbf{r}''(t) \) gives information about the curvature.
In 13.4, we will see that
- \( \mathbf{r}'(t) \) is a velocity vector,
- \( |\mathbf{r}'(t)| \) is the speed, and
- \( \mathbf{r}''(t) \) is an acceleration vector.

We also define
\[
\int \mathbf{r}(t) \, dt = \langle \int x(t) \, dt, \int y(t) \, dt, \int z(t) \, dt \rangle.
\]
Morale, do derivatives and integral component-wise.

Ex: Consider \( \vec{r}(t) = \langle t, \cos(2t), \sin(2t) \rangle \).

(a) Find \( \vec{r}'(t) \), \( |\vec{r}'(t)| \), and \( \vec{r}''(t) \).

(b) Find \( \vec{r}(\pi/4) \) and \( \vec{r}'(\pi/4) \).

(c) Give the equation for the tangent line at \( t = \pi/4 \)
5. Arc Length
The length of a curve from \( t = a \) to \( t = b \) is given by
\[
\int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt
\]
\[
= \int_{a}^{b} |\vec{r}'(t)| \, dt
\]
(Note: 2D is same without the \( z'(t) \)).
We call this **arc length**.
The arc length from 0 to \( u \) is often written as
\[
s(u) = \int_{0}^{u} |\vec{r}'(t)| \, dt
\]
We call this the **arc length function**.