Math 126 Spring 2010
Work with projections, dot products, and cross-products.

1. Decide for each expression below whether it is a vector (V), a scalar (S), or nonsense (N). Note that a, b, u, and v are vectors, while c and d are scalars.

(SEE ATTACHED FOR MORE DETAILED SOLUTIONS)

Circle one:

(a) \(a \cdot (u - cv)\) V S N
(b) \(a \cdot (b + c)\) V S N
(c) \((c + d) \cdot a\) V S N
(d) \(uv\) V S N
(e) \(\frac{a}{c}\) V S N
(f) \(\frac{c}{a}\) V S N
(g) \((a \cdot b) \times u\) V S N
(h) \(a \times (b \times u)\) V S N
(i) \(a \cdot (b \times u)\) V S N
(j) \((ca) \times b\) V S N
(k) \(c(a \cdot b)(u \times v)\) V S N

2. Determine whether each of the following is true or false. If it is true, prove it. If it is false, give a counterexample. Note that a and b are vectors and c is a scalar.

(a) Suppose \(a \cdot b = 0\). Then it must be true that at least one of a or b must be the zero vector. **FALSE!!!**
(b) Suppose \(ca = 0\). Then it must be true that either \(c = 0\) or \(a = 0\) (or both). **TRUE!**

3. Suppose that a and b are nonzero vectors.

(a) Show by examples that \(\text{comp}_ba\) and \(\text{comp}_ab\) can be the same and can be different. What conditions on a and b will guarantee they are the same?

(b) Your friend who skips class frequently says, “I’m confused. Isn’t \(a \cdot b = b \cdot a\)? If that is true, how can \(\text{comp}_ba\) and \(\text{comp}_ab\) be different?” What is your answer?

(c) Show by examples that \(\text{proj}_a b\) and \(\text{proj}_b a\) can be the same and can be different. What conditions on a and b will guarantee they are the same?
Worksheet 2a Notes and Solutions

See page 1 for answers.

Note we have defined:

**Sum/Difference** \[ \mathbf{u} + \mathbf{v} = \mathbf{w} \quad \text{and} \quad \mathbf{u} - \mathbf{v} = \mathbf{w} \]

**Scalar Multiple** \[ c \mathbf{u} = \mathbf{v} \quad \text{No symbol here!} \]

**Dot Product** \[ \mathbf{u} \cdot \mathbf{v} = c \]

**Cross Product** \[ \mathbf{u} \times \mathbf{v} = \mathbf{w} \]

Specific comments:

(a) \[ \frac{\mathbf{a} \cdot (\mathbf{u} - c \mathbf{v})}{\text{vector}} = \frac{\text{vector}}{\text{vector}} = \text{Scalar} \]

(b) \[ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) \quad \text{Nonsense} \]

(c) \[ \left( \frac{c + d}{\text{vector}} \right) \cdot \mathbf{a} \quad \text{This "DOT" symbol is reserved for dot products of two vectors; so this is nonsense. For a scalar product, put no symbol.} \]

(d) \[ \mathbf{u} \cdot \mathbf{v} \quad \text{Nonsense, we did not define this.} \]

(e) \[ \frac{c}{\text{vector}} \quad \text{Some symbol is needed ("\cdot" or "\times"}) \]

(f) \[ \frac{c}{\text{vector}} \quad \text{Scalar vector} \]

(g) \[ \frac{(\mathbf{a} \cdot \mathbf{b})}{\text{vector}} \times \mathbf{u} = \text{Nonsense} \]

(h) \[ \frac{\mathbf{a}}{\text{vector}} \times \left( \frac{\mathbf{b} \times \mathbf{u}}{\text{vector}} \right) = \text{Vector} \]

(i) \[ \frac{\mathbf{a}}{\text{vector}} \cdot \left( \frac{\mathbf{b} \times \mathbf{u}}{\text{vector}} \right) = \text{Scalar} \]
(i) \((c\vec{a}) \times \vec{b} = \text{vector}\)

(k) \(c\frac{(\vec{a} \cdot \vec{b})}{(\vec{u} \times \vec{v})} = \text{vector}\)

No scalar (good)

2 (a) TRUE or FALSE:
If \(\vec{a} \cdot \vec{b} = 0\), then \(\vec{a} = \vec{0}\) or \(\vec{b} = \vec{0}\).

FALSE!!!

Remember \(\vec{a} \cdot \vec{b} = 0 \iff \vec{a}\) and \(\vec{b}\) are orthogonal, that does not mean one of them must be zero.

Counter-example: \(\vec{a} = \langle 1, -2 \rangle, \vec{b} = \langle 2, 1 \rangle\)
\(\vec{a} \cdot \vec{b} = 0\) (hypothesis true)
and \(\vec{a}\) is not the zero vector \((\text{conclusion false})\)
and \(\vec{b}\) is not the zero vector
Hence the statement is not always true. (So it is a false statement)

(b) TRUE or FALSE:
If \(c\vec{a} = \vec{0}\), then \(c = 0\) or \(\vec{a} = \vec{0}\).

TRUE!

proof Assume \(c\vec{a} = \vec{0}\).
Labeling the components of \(\vec{a} = \langle a_1, a_2, a_3 \rangle\)
By definition of scalar multiplication
\(c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle\)
Since we are assuming this is the zero vector, we have \(ca_1 = 0\) and \(ca_2 = 0\) and \(ca_3 = 0\).
So either \(c = 0\), or if it is not, \(a_1/a, a_2/a\) are all zero. Thus, \(c = 0\) or \(\vec{a} = \vec{0}\).
3. (a) \( \text{comp}_x(b) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} \) and \( \text{comp}_y(a) = \frac{\mathbf{a} \cdot \mathbf{y}}{||\mathbf{y}||} \)

So \( \text{comp}_x(b) = \text{comp}_y(a) \Leftrightarrow \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} = \frac{\mathbf{a} \cdot \mathbf{y}}{||\mathbf{y}||} \) same

which can only happen

when i) \( \mathbf{a} \cdot \mathbf{b} = 0 \)

or ii) \( ||\mathbf{a}|| = ||\mathbf{y}|| \).

Thus, \( \text{comp}_x(b) = \text{comp}_y(a) \) exactly when \( \mathbf{a} \cdot \mathbf{b} = 0 \) (i.e. \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal)

or \( ||\mathbf{a}|| = ||\mathbf{y}|| \) (i.e. \( \mathbf{a} \) and \( \mathbf{b} \) have the same length)

Otherwise, the component projections are different.

(b) Remind your friend that component projections also depend on the vector length of the vector you are projecting onto. Thus, give them an example.

(c) \( \text{proj}_x(b) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{b}||} \mathbf{a} \), \( \text{proj}_y(x) = \frac{\mathbf{a} \cdot \mathbf{y}}{||\mathbf{y}||} \mathbf{y} \)

Observe, for these vectors to be equal they must

1. POINT IN THE SAME DIRECTION
2. HAVE THE SAME LENGTH

1. CAN ONLY HAPPEN IF \( \mathbf{a} \) and \( \mathbf{b} \) were already in the same direction. Thus, \( \mathbf{a} \) and \( \mathbf{b} \) must be parallel in the same direction.

2. CAN ONLY HAPPEN IF \( \text{comp}_x(b) = \text{comp}_y(x) \) so \( ||\mathbf{a}|| = ||\mathbf{b}|| \) (from above since not orthogonal)