Taylor Polynomials, Taylor Series, and Final Review
This worksheet is designed to help you to start thinking about the final and to help sort out your understanding of Taylor polynomials. **In small groups discuss the first four questions.** Hand in your work for the first four questions to get credit for your worksheet, then keep this handout for studying. If time permits, discuss questions 5, 6, and 7 with your TA. Most of these questions are from old midterms and final exams. This is not a comprehensive list of topics and this should not be your only source of studying, I just wanted to give you a few old exam questions to start your studying.

1. Find the 2nd-degree Taylor polynomial, \( T_2(x) \) for the function \( f(x) = \ln(\ln x) \) based at \( x = e \).

2. Consider the function \( f(x) = \sin \left( \frac{\pi x}{6} \right) \).
   
   (a) Find \( T_2(x) \), the second order Taylor polynomial for \( f(x) \) centered at \( a = 1 \).
   
   (b) Use Taylor’s inequality to find an upper bound on \( |f(1.1) - T_2(1.1)| \).

3. (a) Find the quadratic approximation, \( T_2(x) \), based at \( b = 1 \), for the function \( f(x) = x \ln x \).

   (b) Use \( T_2(x) \) to approximate \( 0.9 \ln(0.9) \).

   (c) Using Taylor’s inequality, estimate the error of the approximation you obtained in (b).

4. Consider the function \( f(x) = x^3 + x \).
   
   (a) Find the second Taylor polynomial \( T_2 \) of \( f \) based at \( b = 1 \).

   (b) Use Taylor’s inequality to find an interval \( J \) around \( b \) such that the error \( |T_2(x) - f(x)| \) is less than 0.001 for all \( x \) in \( J \).

5. Approximate the integral
   \[
   \int_0^2 \sin(x^2) \, dx
   \]
   by using the first four non-zero terms of a Taylor series. Given a decimal approximation of your result.

6. Write out the first four terms of the Taylor series for the function \( f(x) = \frac{1}{1+5x} + \frac{1}{3+x} \).

7. Give the coefficient on \( x^{11} \) in the Taylor series for \( f(x) = x^3 e^{x^2} \) based at \( b = 0 \).

8. Let \( f(x) = x^3 \cos(5x^2) \). Write down the Taylor series about \( a = 0 \) for the indefinite integral \( \int f(x) \, dx \).

9. Consider the function \( f(x) = \ln(3 + 2x^2) \).
   
   (a) Compute \( f'(x) \) and find its Taylor series centered at zero.

   (b) Use part (a) to find the Taylor series centered at zero for \( f(x) \). (**Hint:** What is \( f(0) \)?)

   (c) What is the radius of convergence of the series you found in part (b)?
10. Find a vector $v$ which satisfies both of the following conditions:
   (i) $v$ is orthogonal to $\langle 2, 1, 4 \rangle$,
   (ii) the cross product of $v$ and $\langle 1, 2, 0 \rangle$ equals $\langle 2, -1, 0 \rangle$.

11. Let $L_1$ be the line given by the parametric equations
    
    $$x = 2t, y = 0, z = 4 - 4t,$$
    
    and let $L_2$ be the line given by the parametric equations
    
    $$x = 2 - 2u, y = 3u, z = 0.$$

   (a) Find the point of intersection of $L_1$ and $L_2$.
   (b) Find an equation of the plane that contains both $L_1$ and $L_2$. Give your answer in the form $ax + by + cz = d$.

12. Find the parametric equations for the line that is the intersection of the plane

    $$x + y + 2z = 1$$

    and the plane

    $$3x - y + 4z = 1.$$

13. Find an equation for the plane through the origin that is perpendicular to the planes $5x - y + z = 1$ and $2x + 2y - 3z = 2$.

14. Suppose the trajectory of a particle is given by

    $$\mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t \mathbf{k}.$$ 

    Calculate the magnitude of the normal component of the acceleration experienced by the particle at $t = 1$.

15. Consider the space curve represented by the vector function $\mathbf{r}(t) = (\cos(t), \cos(t), \sqrt{2} \sin(t))$, where $0 \leq t \leq 2\pi$.

   (a) Compute $\mathbf{r}'(t)$.
   (b) Reparametrize the curve with respect to the arclength.
   (c) Let $P = (1/2, 1/2, \sqrt{3}/2)$. Find the following.
      i. A parametrization of the tangent line for the curve at $P$.
      ii. The curvature of the curve at $P$.
      iii. An equation of the osculating plane for the curve at $P$.
      iv. An equation of the normal plane for the curve at $P$.

16. The position function of a spaceship is $\mathbf{r}(t) = (3 + t, 2 + \ln t, 7 + t^2)$ and the coordinates of the space station are $(7, 5, 14)$. The captain wants the spaceship to coast into the space station. When should the engines be turned off? (That is, we want the value of $t$ for which the tangent line with intersect $(7,5,14)$).

17. Find the vector function $\mathbf{r}(t)$ such that the acceleration is $\mathbf{a}(t) = \mathbf{i} - 12t^2 \mathbf{j} + 2t \mathbf{k}$ and the initial position and velocity are given by $\mathbf{r}(0) = \mathbf{i} + k$ and $\mathbf{v}(0) = 2 \mathbf{j}$. 
18. Consider the function \( f(x, y) = e^{3x+5y-1} \).

(a) Sketch the level sets of \( f \), \( f(x, y) = k \), for \( k = e^{-1} \) and \( k = 1 \). What are the level sets if \( k \leq 0 \)?

(b) Calculate the partial derivatives \( f_x \) and \( f_y \).

(c) Write an equation for the tangent plane to the graph of \( f(x, y) \) at the point \( (2, -1, 1) \).

(d) Use the linear approximation for \( f \) at \( (2, -1) \) to estimate the value \( f(1.8, -0.9) \).

19. Find and classify all the critical points of the function \( f(x, y) = x^3 + y^2 + 2xy \).

20. Integrate the function \( f(x, y) = x + y \) over the region bounded by \( x + y = 2 \) and \( y^2 - 2y - x = 0 \).

21. (a) Reverse the order of integration for the integral

\[ \int_0^4 \int_{\sqrt{x}}^2 xy \, dy \, dx. \]

(b) Evaluate the integral in part (a). You may integrate either in the original order or in the reversed order.

22. Evaluate the following iterated integral:

\[ \int_0^3 \int_0^{9-x^2} \frac{xe^{3y}}{9-y} \, dy \, dx \]

23. Find the volume of the region between the cylinders \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \) and bounded above by \( z = y^2 \) and bounded below by \( z = 0 \).