Taylor Polynomials Overview

We found that we can approximate functions $f(x)$ with polynomials based at $x = b$ in the following way.

$$T_1(x) = \sum_{k=0}^{1} \frac{f^{(k)}(b)}{k!} (x - b)^k = f(b) + f'(b)(x - b).$$

$$T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(b)}{k!} (x - b)^k = f(b) + f'(b)(x - b) + \frac{f''(b)}{2!}(x - b)^2.$$

$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(b)}{k!} (x - b)^k = f(b) + \cdots + \frac{f'''(b)}{3!}(x - b)^3.$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!} (x - b)^k = f(b) + \cdots + \frac{f^{(n)}(b)}{n!}(x - b)^n.$$

**Taylor inequalities**

And we found that we can get a bound on the error in the following way.

$$\text{ERROR} = |f(x) - T_1(x)| \leq \frac{M}{2!} |x - b|^2$$

, where $|f''(x)| \leq M$.

$$\text{ERROR} = |f(x) - T_2(x)| \leq \frac{M}{3!} |x - b|^3$$

, where $|f'''(x)| \leq M$.

$$\text{ERROR} = |f(x) - T_3(x)| \leq \frac{M}{4!} |x - b|^4$$

, where $|f^{(4)}(x)| \leq M$.

$$\text{ERROR} = |f(x) - T_n(x)| \leq \frac{M}{(n + 1)!} |x - b|^{n+1}$$

, where $|f^{(n+1)}(x)| \leq M$.

We asked the following error questions:

1. Given a fixed $n$ and a fixed interval, find the error bound.

2. Given a fixed $n$ and an error, find an interval with an error bound less than the given error.

3. Given a fixed interval and an error, find a number $n$ with an error bound less than the given error.
Taylor Series Overview

Then we started looking for patterns in the Taylor series for some of our standard functions. We found:

\[
e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots , \text{ for all } x.
\]

\[
\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots , \text{ for all } x.
\]

\[
\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots , \text{ for all } x.
\]

\[
\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots , \text{ for } -1 < x < 1.
\]

We learned:

1. We can substitute in for \( x \) in any of these (and in the last case, find the new interval of convergence).

2. We can integrate and differentiate and get a new Taylor series with the same interval of convergence.

Some notable examples include (each of the series below have an interval of convergence of \(-1 < x < 1\)):

\[
-\ln(1-x) = \int_0^x \frac{1}{1-t} \, dt = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots .
\]

\[
\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2} \, dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots .
\]

\[
\frac{1}{(1-x)^2} = \frac{d}{dx} \left( \frac{1}{1-x} \right) = \sum_{k=0}^{\infty} k x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots .
\]

\[
\frac{2}{(1-x)^3} = \frac{d}{dx} \left( \frac{1}{(1-x)^2} \right) = \sum_{k=0}^{\infty} k(k-1) x^{k-2} = 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \cdots .
\]