Facts and Definitions about Three-Dimensional Curves

We will write a three-dimensional parametric curve in either of the equivalent forms \( x = f(t), y = g(t), z = h(t) \) or \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \). Below we discuss such curves.

1. Visualization: Given a parametric curve in three-dimensions. We can try to visualizing the motion using the following tools
   - Eliminate the parameter to get equations relating \( x, y, \) and \( z \). Then try to visualize the resulting surface over which the motion is occurring.
   - Plot points by choosing values of \( t \) and plotting \( (x, y, z) \).
   - Use the tools and measures below to discuss the motion of a curve at a point.

2. Derivatives and Integrals:
   - \( \mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = \) ‘a vector tangent to the curve at \( t \)’ = velocity vector.
   - \( \mathbf{r}''(t) = \langle f''(t), g''(t), h''(t) \rangle = \) acceleration vector.
   - \( \int \mathbf{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle \).

3. Measurements on the Curve:
   - Arc Length = \( \int_{a}^{b} \left| \mathbf{r}'(t) \right| \,dt = \int_{a}^{b} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \,dt \)
   - Curvature = \( \kappa = \frac{\left| d\mathbf{T}/ds \right|}{\left| \mathbf{r}'(t) \right|} = \frac{\left| \mathbf{T}'(t) \right|}{\left| \mathbf{r}'(t) \right|} = \frac{\left| \mathbf{r}'(t) \times \mathbf{r}''(t) \right|}{\left| \mathbf{r}'(t) \right|^3} \). You can calculate curvature for a 2D curve as well by making the third component zero. For a function of the form \( y = f(x) \) in 2D, the formula for curvature becomes \( \kappa = \frac{|f''(x)|}{\left[ 1 + (f'(x))^2 \right]^{3/2}} \).
   - The tangential and normal components of acceleration will be covered in 13.4 and are given by: \( a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\left| \mathbf{r}'(t) \right|} \) (tangential component) and \( a_N = \frac{\left| \mathbf{r}'(t) \times \mathbf{r}''(t) \right|}{\left| \mathbf{r}'(t) \right|} \) (normal component)

4. Normal Vectors: We define \( \mathbf{T}(t) = \frac{1}{\left| \mathbf{r}'(t) \right|} \mathbf{r}'(t) = \) the unit tangent. And from it we get the following
   \[ \mathbf{T}'(t) = \frac{\mathbf{T}'(t)}{\left| \mathbf{T}'(t) \right|} = \) ‘a normal vector (a vector orthogonal to \( \mathbf{T}(t) \)’
   \[ \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left| \mathbf{T}'(t) \right|} = \) ‘the principal unit normal’
   \[ \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \) ‘the binormal vector (orthogonal to the tangent and unit normal)’

5. Related Planes and Lines
   - The tangent line to a curve at a given point can be given by using the \( \mathbf{r}'(t) \) as the direction vector in the equations for the line.
   - The normal plane to a curve at a given point can be given by using \( \mathbf{r}'(t) \) as the normal vector in the equation for a plane.
   - The osculating plane to a curve at a given point can be given by using \( \mathbf{B}(t) \) as the normal vector in the equation for a plane.