1. (12 pts)

(a) Find and simplify an equation for the surface consisting of all points (x, y, z) that are equidistant from the point $(0, 0, 2)$ and the xy-plane. Then give the precise name of this surface.

3

$\sqrt{x^2+y^2+(z-2)^2} = |z|$ $\qquad 4z = x^2+y^2+4$

$x^2+y^2+(z-2)^2 = z^2$

$x^2+y^2+\cancel{z^2}-4z+4=\cancel{z^2}$

Equation: $4z = x^2+y^2+4$

Name: CIRCULAR PARABOLOID

(b) Consider the three point $A(1, 2, 0)$, $B(3, 0, 2)$, $C(4, 1, 0)$.

i. Find the area of the triangle formed by these three points.

3

$\vec{AB} = \langle 2,-2,2\rangle$

$\vec{AC} = \langle 3,-1,0\rangle$

$(0--2)\vec{i} - (0-6)\vec{j} + (-2--6)\vec{k}$

$\langle 2, 6, 4\rangle$

$\text{AREA} = \frac{1}{2}\sqrt{(2)^2+(6)^2+(4)^2}$

$= \frac{1}{2}\sqrt{4+36+16}$

$= \frac{1}{2}\sqrt{56}$

$= \frac{1}{2}\sqrt{4\cdot 14} = \sqrt{14}$

Area = $\frac{1}{2}\sqrt{56} = \sqrt{14}$

ii. Find the equation of the plane containing these three points.

2

$2(x-1)+6(y-2)+4z=0$

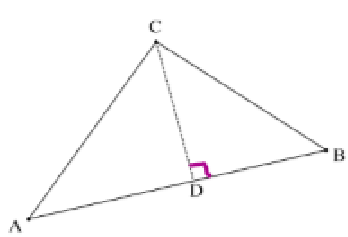
$2x-2+6y-12+4z=0$

$2x+6y+4z=14$ $\rightarrow$ $x+3y+2z=7$

Plane Equation: $2x+6y+4z=14$

iii. The point D is on the line segment from A to B and the line from D to C is perpendicular to that line segment (as shown). Find the distance from A to D and find the coordinates of the point D.

4



$|AD| = \text{comp}_{\langle 2,-2,2\rangle}\langle 3,-1,0\rangle = \frac{6+2+0}{\sqrt{4+4+4}}$

$= \frac{8}{\sqrt{12}}$

$= \frac{4}{\sqrt{3}}$

$\vec{AD} = \frac{8}{\sqrt{12}}\frac{1}{\sqrt{12}}\langle 2,-2,2\rangle$

$= \frac{2}{3}\langle 2,-2,2\rangle$

$= \langle \frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\rangle$

Dist from A to D: $4/\sqrt{3}$

Coordinates D: $(x, y, z) = (1+4/3, 2-4/3, 4/3)$

$(7/3, 2/3, 4/3)$

2. (14 pts)

4

(a) Determine whether each statement is true or false in $\mathbb{R}^3$.
(Put "×" in the circle next to your choice)

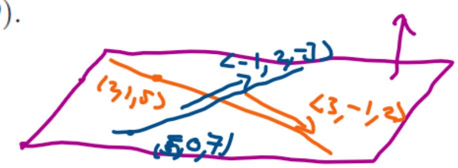
i. ☒ TRUE ◯ FALSE : Two lines perpendicular to a given plane are parallel.

ii. ◯ TRUE ☒ FALSE : Two planes parallel to a given line are parallel.

4

(b) Find the equation for the plane containing the intersecting lines
$L_1 : x = 2 + 3t, y = 1 - t, z = 5 + 2t$ and $L_2 : x = 5 - u, y = 2u, z = 7 - 3u$
(Simplify your answer into the form $Ax + By + Cz = D$).

POINT: (2,1,5) OR (5,0,7)



NORMAL: $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ -1 & 2 & -3 \end{vmatrix}$

$= (3-4)\vec{i} - (-9--2)\vec{j} + (6-1)\vec{k}$

$= \langle -1, 7, 5 \rangle$

$-(x-2) + 7(y-1) + 5(z-5) = 0$

$-x + 2 + 7y - 7 + 5z - 25 = 0$

$-x + 7y + 5z = 30$

Plane Equation: $-x + 7y + 5z = 30$

6

(c) Find parametric equations for the line of intersection of the two planes $2x - y + 3z = 10$ and $x - y + z = 2$.

① $2x - y + 3z = 10$

② $x - y + z = 2$

$x + 0 + 2z = 8$

$x = 0 \Rightarrow z = 4 \Rightarrow 0 - y + 4 = 2$, $y = 2$

$P(0,2,4)$

$z = 0 \Rightarrow x = 8 \Rightarrow 8 - y + 0 = 2$, $y = 6$

$Q(8,6,0)$

$\vec{PQ} = \langle 8, 4, -4 \rangle$

ALSO COULD COMPUTE DIRECTION VECTOR VIA

$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 1 & -1 & 1 \end{vmatrix}$

$= (-1 - -3)\vec{i} - (2-3)\vec{j} + (-2-1)\vec{k}$

$= \langle 2, 1, -1 \rangle$

ALSO CORRECT → $x = 2t, y = 2 + t, z = 4 - t$

* DIRECTION VECTOR MUST BE PARALLEL TO $\langle 2, 1, -1 \rangle$

ONE ANSWER →

Line Equations: $x = 8t, y = 2 + 4t, z = 4 - 4t$

3. (12 pts) Consider curve given by $\mathbf{r}_1(t) = \langle 2t+1, -3t, t^3 - t^2 + 4\rangle$.

6

(a) Find equations for the tangent line to the curve at the point $(5, -6, 8)$. And give the point of intersection of this tangent line with the plane $x + y + z = 42$.

$x = 5 = 2t+1$, $y = -6 = -3t$, $z = 8 = t^3 - t^2 + 4 \Rightarrow t = 2$

$\vec{r}_1{}'(t) = \langle 2, -3, 3t^2 - 2t\rangle$

$\vec{r}_1{}'(2) = \langle 2, -3, 12-4\rangle = \langle 2, -3, 8\rangle$

$x = 5 + 2v$
$y = -6 - 3v$
$z = 8 + 8v$

$x + y + z = 42$

$(5+2v) + (-6-3v) + (8+8v) = 42$

$7 + 7v = 42$
$7v = 35$
$v = 5$

$x = 5 + 2(5) = 15$
$y = -6 - 3(5) = -21$
$z = 8 + 8(5) = 48$

Intersection point $(x, y, z) =$ $(15, -21, 48)$

6

(b) Find the angle of intersection (to the nearest degree) of the given curve $\mathbf{r}_1(t)$ and the second curve $\mathbf{r}_2(u) = \langle 4+u, -6, 4\sqrt{u^2+3}\rangle$.

$2t + 1 = 4 + u$, $-3t = -6$, $t^3 - t^2 + 4 = 4\sqrt{u^2+3}$
$t = 2$; $t^3 - t^2 + 4 = 8$ CHECK!; $4\sqrt{u^2+3} = 8$ ✓
$2t = 3 + u$
$4 = 3 + u$
$u = 1$

$\vec{r}_1{}'(2) = \langle 2, -3, 8\rangle$

$\vec{r}_2{}'(u) = \langle 1, 0, \frac{4 \cdot 2u}{2\sqrt{u^2+3}}\rangle$

$\vec{r}_2{}'(1) = \langle 1, 0, 2\rangle$

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$\Rightarrow \theta = \cos^{-1}\left(\frac{2+0+16}{\sqrt{4+9+64}\sqrt{1+0+4}}\right) = \cos^{-1}\left(\frac{18}{\sqrt{77}\sqrt{5}}\right)$

$\approx 23.46°$

Angle = 23 degrees

4. (12 pts)

6

(a) Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2e^{2t}\mathbf{i} + \sec^2(t)\mathbf{j} + t\sin(t^2)\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

$\vec{r}(t) = \langle e^{2t} + C_1, \tan(t) + C_2, -\frac{1}{2}\cos(t^2) + C_3 \rangle$

$e^0 + C_1 = 1 \Rightarrow C_1 = 0$

$\tan(0) + C_2 = 2 \Rightarrow C_2 = 2$

$-\frac{1}{2}\cos(0) + C_3 = 3 \Rightarrow C_3 = 3 + \frac{1}{2} = \frac{7}{2}$

$\int t\sin(t^2)\,dt = \int \sin(u)\frac{1}{2}\,du = -\frac{1}{2}\cos(u) + C_3$

$u = t^2$

$du = 2t\,dt$

$\frac{1}{2t}du = dt$

$\mathbf{r}(t) =$ $\langle e^{2t}, \tan(t) + 2, -\frac{1}{2}\cos(t^2) + \frac{7}{2} \rangle$

6

(b) Find the value of x at which $f(x) = e^x$ has maximum curvature.

$f'(x) = e^x, \quad f''(x) = e^x$

$$k(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}$$

$$k'(x) = \frac{(1 + e^{2x})^{3/2}e^x - e^x \frac{3}{2}(1 + e^{2x})^{\frac{1}{2}} 2e^{2x}}{(1 + e^{2x})^3} \stackrel{?}{=} 0$$

$$e^x(1 + e^{2x})^{\frac{1}{2}}\left((1 + e^{2x})^1 - 3e^{2x}\right) = 0$$

CAN'T BE ZERO

$1 - 2e^{2x} = 0$

$e^{2x} = \frac{1}{2}$

$2x = \ln\left(\frac{1}{2}\right)$

$x = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln(2) = \ln\left(\sqrt{\frac{1}{2}}\right)$

$x =$ $-\frac{1}{2}\ln(2)$