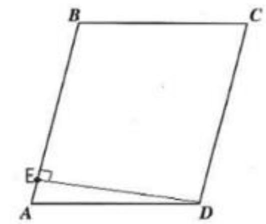
13

1. (~~14~~ pts) The parallelogram below has corners at the points A, B, C and D. The point E is discussed in the last question on this page. You are given the following information:



- The location of the points $A(1, 2, 3)$ and $B(2, 3, 5)$.
- The vector $\overrightarrow{AD} = \langle 3, -1, 0 \rangle$

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(a) Give the (x, y, z) coordinates of the point C.

MANY WAYS TO GET THIS.
THE EASIEST IS PROBABLY TO REALIZE THAT
$\overrightarrow{BC} = \overrightarrow{AD} = \langle 3, -1, 0 \rangle$ SO TO GET FROM B TO C WE DO $\Delta x = 3$, $\Delta y = -1$, $\Delta z = 0$

$(2+3, 3-1, 5+0)$

$(x, y, z) = \boxed{(5, 2, 5)}$

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(b) Find the area of the parallelogram and give the equation of the plane containing this parallelogram.

$\vec{i} \quad \vec{j} \quad \vec{k}$

$\overrightarrow{AB} = \langle 1, 1, 2 \rangle$

$\overrightarrow{AD} = \langle 3, -1, 0 \rangle$

$= (0 - -2)\vec{i} - (0-6)\vec{j} + (-1-3)\vec{k}$

$= \langle 2, 6, -4 \rangle$

$\sqrt{4+36+16} = \sqrt{56}$

Area = $\sqrt{56}$

Plane Equation = $2(x-1) + 6(y-2) - 4(z-3) = 0$

3

(c) Find the angle $\angle BAD$, which is the angle at the vertex A.
(*Give your final answer in degrees, rounded to the nearest degree*)

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{AB} \cdot \overrightarrow{AD}}{|\overrightarrow{AB}||\overrightarrow{AD}|}\right) = \cos^{-1}\left(\frac{3-1+0}{\sqrt{6}\sqrt{10}}\right) \approx 75.04^\circ$$

$\simeq 1.3096$ RADIANS

$$= \cos^{-1}\left(\frac{2}{\sqrt{60}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{15}}\right)$$

Angle = $\boxed{75}$ degrees

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(d) The point E is on the line segment from A to B and the line from E to D is perpendicular to the line segment from A to B (as shown). Find the vector $\overrightarrow{DE}$.

$$\overrightarrow{AE} = \text{proj}_{\overrightarrow{AB}} \overrightarrow{AD} = \frac{\overrightarrow{AD} \cdot \overrightarrow{AB}}{\overrightarrow{AB} \cdot \overrightarrow{AB}} \overrightarrow{AB} = \frac{2}{6}\langle 1, 1, 2 \rangle = \left\langle \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3} \right\rangle$$

$$\overrightarrow{DE} = \overrightarrow{AE} - \overrightarrow{AD} = \left\langle \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3} \right\rangle - \langle 3, -1, 0 \rangle$$

$\overrightarrow{DE}$-vector: $\boxed{\left\langle -\tfrac{8}{3}, \tfrac{4}{3}, \tfrac{2}{3} \right\rangle}$

2. (12 pts)

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(a) Determine whether each statement is true or false in $\mathbb{R}^3$.
(Put "$\times$" in the circle next to your choice) CAN BE SKEW

i. ○ TRUE ⊗ FALSE : Two different lines parallel to a given plane must be parallel.

ii. ⊗ TRUE ○ FALSE : Two different planes perpendicular to a given line must be parallel. TRUE

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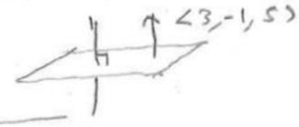
(b) Consider the line L_1 given by $x = 9 + t$, $y = 4 + 2t$, $z = 1 - 5t$. A second line, L_2, is perpendicular to the plane $3x - y + 5z = 30$ and intersects this plane at its z-intercept. Give parametric equations for the line L_2 and find the intersection of the two lines L_1 and L_2 (*if they do not intersect, then write DNE*).



Z - INTERCEPT : $(0,0,6)$

L_2 : $x = 0 + 3u$
$y = 0 - u$
$z = 6 + 5u$

SOLVE ① $9 + t = 3u$
② $4 + 2t = -u \rightarrow u = -4 - 2t$
③ $1 - 5t = 6 + 5u$

$9 + t = -12 - 6t$
$7t = -21$
$t = -3 \rightarrow (x,y,z) = (6,-2,16)$
$u = 2 \rightarrow (x,y,z) = (6,-2,16)$

Line Equations for L_2: $x = 3u$, $y = -u$, $z = 6 + 5u$

Intersection of L_1 and L_2: $(x, y, z) =$ $(6,-2,16)$

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(c) Find the equation of the plane that passes through the point $(3, 2, 1)$ and contains the line of intersection of the two planes $2x + y + 5z = 9$ and $x - y + z = 3$.

① $2x + y + 5z = 9$
② $x - y + z = 3$

$3x + 6z = 12$
$x + 2z = 4$

$C(3,2,1)$

2 POINTS

$A(0,-1,2) \rightarrow 0 - y + 2 = 3 \Rightarrow y = -1$

$B(4,1,0) \rightarrow 4 - y + 0 = 3 \Rightarrow y = 1$

MAY ALSO USE
$\langle 2,1,5\rangle \times \langle 1,-1,1\rangle$
$= \langle 6,3,-3\rangle$
IN PLACE OF $\overrightarrow{AB}$,
IN WHICH CASE WE GET $\langle 6,-3,9\rangle$ HERE
WHICH IS ALSO CORRECT.

$\vec{i} \quad \vec{j} \quad \vec{k}$

$\overrightarrow{AB} = \langle 4, 2, -2\rangle$

$\overrightarrow{AC} = \langle 3, 3, -1\rangle$

$= (-2 - -6)\vec{i} - (-4 - -6)\vec{j} + (12 - 6)\vec{k}$

$= \langle 4, -2, 6\rangle$

Plane Equation: $4(x-3) - 2(y-2) + 6(z-1) = 0$

3. (12 pts) Consider curve given by $\mathbf{r}_1(t) = \langle \sqrt{4t+1}, e^{(t^2-4)}, t^3 \rangle$.

6

(a) Find parametric equations for the tangent line to the curve at the point $(3, 1, 8)$. And give the point of intersection of this tangent line with the plane $3x - 2y + z = 33$.

$3 = \sqrt{4t+1}, \quad 1 = e^{(t^2-4)} \quad 8 = t^3 \Rightarrow t = 2$

$\vec{r}(2) = \langle \sqrt{9}, e^0, 2^3 \rangle = \langle 3, 1, 8 \rangle \leftarrow$ LOCATION

$\vec{r}\,'(t) = \left\langle \frac{4}{2\sqrt{4t+1}}, 2t\,e^{(t^2-4)}, 3t^2 \right\rangle$

$\vec{r}\,'(2) = \left\langle \frac{2}{3}, 4, 12 \right\rangle \leftarrow$ TANGENT DIRECTION

$$\left.\begin{aligned} x &= 3 + \tfrac{2}{3}u \\ y &= 1 + 4u \\ z &= 8 + 12u \end{aligned}\right\} \longrightarrow \overbrace{9 + 2u}^{3x} \overbrace{- 2 - 8u}^{-2y} + \overbrace{8 + 12u}^{+z} = 33$$

$6u = 18$

$u = 3$

Intersection point $(x, y, z) =$ $(5, 13, 44)$

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(b) Find the arc length of the curve of intersection of the elliptical cylinder $4x^2 + y^2 = 4$ and the plane $z = \sqrt{3}x$. *(Parameterize AND compute the arc length integral)*

$$\left.\begin{aligned} x &= \cos(t) \\ y &= 2\sin(t) \end{aligned}\right\} \quad 4x^2 + y^2 = 4\cos^2(t) + 4\sin^2(t) = 4 \checkmark$$

$z = \sqrt{3}\cos(t)$

$$\int_0^{2\pi} \sqrt{(-\sin(t))^2 + (2\cos(t))^2 + (-\sqrt{3}\sin(t))^2}\, dt$$

$$\int_0^{2\pi} \sqrt{\sin^2(t) + 4\cos^2(t) + 3\sin^2(t)}\, dt$$

$$\int_0^{2\pi} \sqrt{4(\cos^2(t) + \sin^2(t))}\, dt$$

$$= \int_0^{2\pi} 2\, dt = 2t \Big|_0^{2\pi} = 4\pi$$

Length = 4π ~~degrees~~

4. (~~12~~ 13 pts)

(a) Find and simplify an equation for the surface consisting of all points P whose distance to the y-axis is 5 times the distance to the plane $y = 1$. Then give the precise name of this surface.

3

"DIST FROM (x,y,z) TO $(0,y,0)$" $= 5 \cdot$ "DIST FROM (x,y,z) TO $(x,1,z)$"

$$\sqrt{(x-0)^2+(y-y)^2+(z-0)^2} = 5 \cdot \sqrt{(x-x)^2+(y-1)^2+(z-z)^2}$$

$$\sqrt{x^2+z^2} = 5\sqrt{(y-1)^2}$$

$$\boxed{x^2+z^2 = 25(y-1)^2}$$

Equation: $x^2+z^2=25(y-1)^2$

Name: CONE (OR CIRCULAR CONE)

(b) Find all constants, a, so that the curve $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), at^2\rangle$ has a curvature of $\kappa(0) = 1$ at $t = 0$.

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$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), 2at\rangle \Rightarrow \vec{r}\,'(0) = \langle 0, 2, 0\rangle \quad |\vec{r}\,'(0)| = 2$

$\vec{r}\,''(t) = \langle -2\cos(t), -2\sin(t), 2a\rangle \rightarrow \vec{r}\,''(0) = \langle -2, 0, 2a\rangle$

$\vec{i} \quad \vec{j} \quad \vec{k}$

$= (4a-0)\vec{i} - (0-0)\vec{j} + (0--4)\vec{k}$

$= \langle 4a, 0, 4\rangle$

$$k(0) = \frac{\sqrt{16a^2+16}}{(2)^3} = \frac{4}{8}\sqrt{a^2+1}$$

$$\frac{1}{2}\sqrt{a^2+1} = 1$$

$$\sqrt{a^2+1} = 2$$

$$a^2+1 = 4 \rightarrow a^2 = 3 \qquad a = \pm\sqrt{3}$$

$a =$ $\pm\sqrt{3}$

(c) Dr. Loveless throws an object into the air. Gravity and wind act on the object as it moves. The acceleration of the object is given by $\mathbf{a}(t) = \langle e^{-t}, 0, -10\rangle$. The initial velocity is $\mathbf{v}(0) = \langle 0, 4, 10\rangle$ and the initial position is $\mathbf{r}(0) = \langle 0, 0, 15\rangle$. ~~The speed of the object at the positive time when it hits the xy-plane. Hint: First find the position functions~~ (~~*Give your final answer rounded to two digits after the decimal.*~~)

5

$\vec{v}(0) = \langle 0, 4, 10\rangle$

$\vec{v}(t) = \langle -e^{-t}+C_1, C_2, -10t+C_3\rangle \Rightarrow -1+C_1=0 \Rightarrow C_1 = 1$

$= \langle -e^{-t}+1, 4, -10t+10\rangle$

$\vec{r}(0) = \langle 0,0,15\rangle$

$\vec{r}(t) = \langle e^{-t}+t+D_1, 4t+D_2, -5t^2+10t+D_3\rangle \qquad 1+0+D_1=0, \; D_1=-1$

$= \langle e^{-t}+t-1, 4t, -5t^2+10t+15\rangle$

$-5t^2+10t+15=0$

$-5(t^2-2t-3)=0$

$-5(t-3)(t+1)=0 \rightarrow t = 3$

~~speed~~ = $\langle -e^{-3}+1, 4, -20\rangle$