WINTER 2018 FINAL QUESTION 7

7 LET $f(x) = x^2 \sin(x^3) + \frac{1}{8-x^3}$

(a) FIND $T_6(x)$ BASED AT $b=0$.
NOTE: FINDING SIX DERIVATIVES WOULD BE TERRIBLE!
USE KNOWN SERIES!

WE ALREADY KNOW

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k \quad -1<t<1 \quad \text{AND}$$
$$\sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} \quad \text{FOR ALL } t$$

SO

$$x^2 \sin(x^3) = x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \overbrace{x^{6k+3}}^{(x^3)^{2k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{6k+5}, \text{ AND}$$

$$\frac{1}{8-x^3} = \frac{1}{8} \frac{1}{1-\frac{x^3}{8}} = \frac{1}{8} \sum_{k=0}^{\infty} \left(\frac{x^3}{8}\right)^k = \frac{1}{8} \sum_{k=0}^{\infty} \frac{x^{3k}}{8^k}$$

THUS,

$$x^2 \sin(x^3) + \frac{1}{8-x^3} = \left(x^5 - \frac{1}{3!}x^{11} + \frac{1}{5!}x^{17} - \cdots\right) + \frac{1}{8}\left(1 + \frac{x^3}{8} + \frac{x^6}{64} + \cdots\right)$$

KEEP EVERYTHING UP TO x^6!!!

$$T_6(x) = x^5 + \frac{1}{8}\left(1 + \frac{x^3}{8} + \frac{x^6}{64}\right)$$
$$= \boxed{\frac{1}{8} + \frac{1}{64}x^3 + x^5 + \frac{1}{512}x^6}$$

(b) $-1 < \frac{x^3}{8} < 1 \Rightarrow -8 < x^3 < 8 \Rightarrow \boxed{-2 < x < 2}$

SPRING 2016 FINAL QUESTION 9

9 LET $F(x) = \int_0^x \frac{t^3}{9+t^2}\,dt$

(a) TAYLOR SERIES BASED AT $b=0$?

WE ALREADY KNOW FROM LECTURE

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad -1<z<1$$

So
$$\frac{t^3}{9+t^2} = \frac{t^3}{9}\cdot\frac{1}{1-\left(-\frac{t^2}{9}\right)} = \frac{t^3}{9}\sum_{k=0}^{\infty}\left(\frac{-t^2}{9}\right)^k \qquad -1<-\frac{t^2}{9}<1$$

$$= \frac{1}{9}t^3\sum_{k=0}^{\infty}\frac{(-1)^k}{9^k}t^{2k} \qquad 9>t^2>-9$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{9^{k+1}}t^{2k+3} \qquad \boxed{-3<t<3}$$

NOW INTEGRATE!

$$F(x) = \int_0^x \sum_{k=0}^{\infty}\frac{(-1)^k}{9^{k+1}}t^{2k+3}\,dt = \sum_{k=0}^{\infty}\frac{(-1)^k}{9^{k+1}}\frac{1}{(2k+4)}t^{2k+4}\Big|_0^x$$

$$\boxed{= \sum_{k=0}^{\infty}\frac{(-1)^k}{9^{k+1}}\frac{1}{(2k+4)}x^{2k+4} = \frac{1}{9}\cdot\frac{1}{4}x^4 - \frac{1}{81}\frac{1}{6}x^6 + \dots}$$

(b) INTERVAL OF CONVERGENCE? $\boxed{-3<x<3}$

(c) $F^{(10)}(0)$ WILL APPEAR IN THE TERM

$\frac{1}{10!}F^{(10)}(0)x^{10}$ AND WE KNOW x^{10} WILL HAPPEN WHEN $10 = 2k+4 \Rightarrow 6 = 2k \Rightarrow k=3$

So
$$\frac{1}{10!}F^{(10)}(0) = \frac{(-1)^3}{9^4}\frac{1}{10} \Rightarrow \boxed{F^{(10)}(0) = \frac{-10!}{9^4\cdot 10}}$$

WINTER 2016 FINAL QUESTION 7

7 LET $f(x) = \ln(1+3x) + x e^{-2x} - \dfrac{4x}{1+5x}$

(a) TAYLOR SERIES BASED AT $b=0$?

WE KNOW FROM LECTURE THAT

$$-\ln(1-t) = \sum_{k=0}^{\infty} \frac{1}{k+1} t^{k+1}, \quad e^t = \sum_{k=0}^{\infty} \frac{1}{k!} t^k, \quad \frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$$

$-1<t<1$ ALL t $-1<t<1$

SO

$$\ln(1-(-3x)) + x e^{-2x} - 4x \frac{1}{1-(-5x)}$$

$$= \left(-\sum_{k=0}^{\infty} \frac{1}{(k+1)} (-3)^{k+1} x^{k+1}\right) + x\left(\sum_{k=0}^{\infty} \frac{1}{k!} (-2)^k x^k\right) - 4x \left(\sum_{k=0}^{\infty} (-5)^k x^k\right)$$

$$= \boxed{\sum_{k=0}^{\infty} \left(\frac{-(-1)^{k+1} 3^{k+1}}{k+1} + \frac{(-1)^k 2^k}{k!} - 4(-1)^k 5^k \right) x^{k+1}}$$

(b) INTERVAL OF CONVERGENCE?

$-1 < -3x < 1 \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$

$-1 < -5x < 1 \Rightarrow -\frac{1}{5} < x < \frac{1}{5}$

BOTH TRUE (OVERLAP) $\boxed{-\frac{1}{5} < x < \frac{1}{5}}$

(c)

STOP!!

$$\int_0^x f(t)\,dt \approx \int_0^x \underbrace{\left(\frac{3}{1} + \frac{1}{1} - 4\right)}_{0} t + \underbrace{\left(\frac{-3^2}{2} - \frac{2}{1} + 20\right)}_{-\frac{9}{2}-2+20} t^2 + \cdots dt$$

$$\approx \int_0^x \left(-\frac{9}{2} + 18\right) t^2\,dt \qquad -\frac{9}{2} + 18 = \frac{27}{2}$$

$$= \frac{27}{2} \cdot \frac{1}{3} t^3 \Big|_0^x = \boxed{\frac{9}{2} x^3}$$

INTEGRATING GIVES x^3

STOPPING HERE

FALL 2017 FINAL QUESTION 8

8 (a) FIND THE TAYLOR SERIES FOR
$f(x) = x^4 \arctan(x^3)$ BASED AT $b=0$.

WE ALREADY KNOW FROM LECTURE THAT

$$\arctan(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k+1} \qquad -1<t<1$$

$(x^3)^{2k+1}$

SO

$$x^4 \arctan(x^3) = x^4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{6k+3} \qquad -1<x^3<1$$

$$-1<x<1$$

$$\boxed{= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{6k+7} = x^7 - \frac{1}{3}x^{13} + \frac{1}{5}x^{19} - \cdots}$$

(b) INTERVAL OF CONVERGENCE

$$\boxed{-1<x<1}$$

(c) $f^{(2017)}(0)$ WILL OCCUR IN THE TERM

$\frac{1}{2017!} f^{(2017)}(0) x^{2017}$ AND WE KNOW THAT x^{2017}

WILL OCCUR WHEN $2017 = 6k+7 \Rightarrow 2010 = 6k \Rightarrow k = 335$

THUS

$$\frac{1}{2017!} f^{(2017)}(0) = \frac{(-1)^{35}}{671} \Rightarrow \boxed{f^{(2017)}(0) = -\frac{(2017)!}{671}}$$

WINTER 2019 FINAL QUESTION 8

$\boxed{8}$ LET $f(x) = \dfrac{x^2}{(1-x)^2}$.

(a) TAYLOR SERIES BASED AT $b=0$?

HINT: $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$

FROM LECTURE WE KNOW

$\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad -1<x<1$, SO

$\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \sum_{k=0}^{\infty} k x^{k-1}$ $\boxed{-1<x<1}$

$\Rightarrow \dfrac{x^2}{(1-x)^2} = x^2 \sum_{k=0}^{\infty} k x^{k-1} = \boxed{\sum_{k=0}^{\infty} k x^{k+1} = 0 + 1x^2 + 2x^3 + 3x^4 + \cdots}$

(b) $f^{(100)}(0)$ WILL OCCUR IN THE TERM

$\dfrac{1}{100!} f^{(100)}(0) x^{100}$ AND WE KNOW THAT x^{100}

WILL OCCUR WHEN $k+1 = 100 \Rightarrow k = 99$.

THUS, $\dfrac{1}{100!} f^{(100)}(0) = 99 \Rightarrow \boxed{f^{(100)}(0) = 99 \cdot (100!)}$

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WINTER 2015 FINAL QUESTION 8

$\boxed{8}$ LET $f(x) = x\arctan(x) - \frac{1}{2}\ln(1+x^2)$

(a) GIVE TAYLOR SERIES BASED AT $b=0$ AND FIRST THREE NON-ZERO TERMS.

WE ALREADY KNOW FROM LECTURE THAT

x

$$\arctan(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k+1} \quad \text{FOR } -1<t<1, \text{ AND}$$

$$-\ln(1-t) = \sum_{k=0}^{\infty} \frac{1}{k+1} t^{k+1} \quad \text{FOR } -1<t<1$$

$-x^2$

ASIDE: $(-x^2)^{k+1} = (-1)^{k+1} x^{2k+2}$

SO

$$x\arctan(x) - \frac{1}{2}\ln(1-(-x^2))$$

$$= x\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}\right) + \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1} x^{2k+2}\right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+2} + \sum_{k=0}^{\infty} \frac{1}{2} \frac{(-1)^{k+1}}{k+1} x^{2k+2}$$

$$= \boxed{\sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2k+1} + \frac{1}{2}\frac{(-1)^{k+1}}{(k+1)}\right) x^{2k+2}}$$

$\frac{1}{4} - \frac{1}{3} = \frac{3}{12} - \frac{4}{12}$

$\frac{1}{5} - \frac{1}{6} = \frac{1}{30}$

$$= \left(\frac{1}{1} + \frac{1}{2}\cdot\frac{-1}{1}\right)x^2 + \left(\frac{-1}{3} + \frac{1}{2}\frac{1}{(2)}\right)x^4 + \left(\frac{1}{5} + \frac{1}{2}\cdot\frac{-1}{(3)}\right)x^6 + \cdots$$

$$= \boxed{\frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{30}x^6 + \cdots}$$

(b) INTERVAL OF CONVERGENCE?

$\left.\begin{array}{l} -1<x<1 \quad \text{AND} \\ -1 < -x^2 < 1 \end{array}\right\} \Rightarrow \boxed{-1<x<1}$

ASIDE: $1 > x^2 \underbrace{> -1}_{\text{NEVER NEGATIVE}}$ $\quad x^2<1$

$\Rightarrow -1<x<1$

FALL 2014 FINAL QUESTION 8

$\boxed{8}$ LET $f(x) = \dfrac{x^2}{x^2 - e^2} + x\sin(\pi x - x)$

(a) TAYLOR SERIES BASED AT $b=0$?

WE ALREADY KNOW

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k \qquad \text{AND} \qquad \sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}$$

$-1 < t < 1$ $\qquad t = x^2/e^2$ $\qquad$ FOR ALL t

SO

$$\frac{x^2}{x^2-e^2} = \frac{-1}{e^2} x^2 \frac{1}{1 - \frac{x^2}{e^2}} = \frac{-1}{e^2} x^2 \sum_{k=0}^{\infty} \left(\frac{x^2}{e^2}\right)^k$$

$$= -\frac{1}{e^2} x^2 \sum_{k=0}^{\infty} \frac{x^{2k}}{e^{2k}} = -\sum_{k=0}^{\infty} \frac{x^{2k+2}}{e^{2k+2}}$$

$$x\sin(\pi x - x) = x\sin((\pi-1)x) = x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}((\pi-1)x)^{2k+1}$$

$$= x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(\pi-1)^{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(\pi-1)^{2k+1} x^{2k+2}$$

THUS, $\boxed{f(x) = \sum_{k=0}^{\infty} \left(-\frac{1}{e^{2k+2}} + \frac{(-1)^k}{(2k+1)!}(\pi-1)^{2k+1} \right) x^{2k+2}}$

$$-1 < \frac{x^2}{e^2} < 1 \Rightarrow -e^2 < x^2 < e^2 \Rightarrow \boxed{-e < x < e}$$

(b) $f^{(674)}(0)$ WILL APPEAR IN THIS TERM

$$\frac{1}{674!} f^{(674)}(0)\, x^{674} \qquad \text{AND}$$

WE KNOW x^{674} OCCURS WHEN $2k+2 = 674 \Rightarrow 2k = 672$
$\Rightarrow k = 336$

THUS,

$$\frac{1}{674!} f^{(674)}(0) = \frac{-1}{e^{674}} + \frac{1}{673!}(\pi-1)^{673}$$

$$\Rightarrow \boxed{f^{(674)}(0) = 674!\left(\frac{-1}{e^{674}} + \frac{(\pi-1)^{673}}{673!} \right)}$$