W'18 Final Prob. 8 – Lecture D

Let $g(x) = \sqrt{3 + x^2}$.

(a) Find the 1st Taylor Polynomial based at b = 1.

(b) Use your answer to approximate the value of $\sqrt{3.25}$.

(c) Use Taylor's inequality to find an upper bound for the error of this approximation.

(a) $g'(x) = \dfrac{\cancel{2}x}{\cancel{2}\sqrt{3+x^2}}$ ← N ← D

$T_1(x) = \underbrace{2}_{g(1)} + \underbrace{\tfrac{1}{2}}_{g'(1)}(x-1)$

$\boxed{\sqrt{3+x^2} \approx 2 + \tfrac{1}{2}(x-1)}$

(b) $\sqrt{3.25} = g\left(\tfrac{1}{2}\right) = \sqrt{3 + \tfrac{1}{4}} \approx 2 + \tfrac{1}{2}\left(\tfrac{1}{2} - 1\right) = 2 + \tfrac{1}{2}\left(-\tfrac{1}{2}\right) = 2 - \tfrac{1}{4} = 1.75$

(c) $\boxed{\text{STEP 1}}$ $g''(x) = \dfrac{DN' - ND'}{D^2} = \dfrac{\sqrt{3+x^2}\cdot 1 - x\dfrac{\cancel{2}x}{\cancel{2}\sqrt{3+x^2}}}{3+x^2} \cdot \dfrac{\sqrt{3+x^2}}{\sqrt{3+x^2}}$ ← TO SIMPLIFY

$g''(x) = \dfrac{(3+x^2) - x^2}{(3+x^2)^{3/2}} = \dfrac{3}{(3+x^2)^{3/2}} \Rightarrow |g''(x)| = \dfrac{3}{(3+x^2)^{3/2}} \le \dfrac{3}{(3.25)^{3/2}}$

BIGGEST WHEN DENOM. SMALLEST

$\tfrac{1}{2} \le x \le 1$

APPROX ↑ ↑ BASED

$\boxed{\text{STEP 2}}$

$\text{ERROR} = |g(x) - T_1(x)| \le \dfrac{}{2!}|\overset{1/2}{x} - 1|^2 \le \dfrac{1}{2}\dfrac{3}{(3.25)^{3/2}}\left(\dfrac{1}{2}\right)^2 = \boxed{\dfrac{3}{8}\dfrac{1}{(3.25)^{3/2}}}$

Sp'16 Final Problem 8 – Lecture C

Let $f(x) = e^{x^2}$.

(a) Find the second Taylor Polynomial based at b = 1.

(b) Find an upper bound for $|T_2(x) - f(x)|$ on the interval [0,2].

(c) What is the ***smallest*** value of $|T_2(x) - f(x)|$ on the interval [0,2]?

(a)

$$f(x) = e^{x^2}$$

$$f'(x) = 2xe^{x^2}$$

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = (2+4x^2)e^{x^2}$$

$$\boxed{\begin{aligned} T_2(x) &= \underbrace{e} + \underbrace{2e}(x-1) + \frac{1}{2!}\underbrace{6e}(x-1)^2 \\ T_2(x) &= e\left[1 + 2(x-1) + 3(x-1)^2\right] \end{aligned}}$$

(b) $\boxed{\text{STEP 1}}$

$$f'''(x) = 8xe^{x^2} + (2+4x^2)2xe^{x^2}$$

$$= 8xe^{x^2} + (4x+8x^3)e^{x^2}$$

$$= (2 + 1 + 2x^2)4xe^{x^2}$$

$$|f'''(x)| = |\underbrace{(3+2x^2)}_{\le 11}\underbrace{4x}_{\le 8}\underbrace{e^{x^2}}_{\le e^4}| \le 88e^4$$

FOR $0 \le x \le 2$

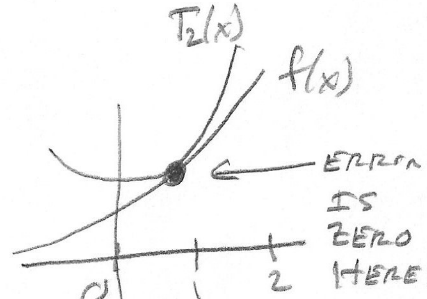
$\boxed{\text{STEP 2}}$

$$\text{ERROR} = |T_2(x) - f(x)| \le \frac{88e^4}{3!}|x-1|^3$$

1

$$\frac{88}{6}e^4 = \boxed{\frac{44}{3}e^4}$$

(c)



SAME

$$T_2(1) - f(1) = 0$$

$$\boxed{\text{SO THE } \underline{\text{SMALLEST}}\ |T_2(x) - f(x)| \text{ CAN BE IS } 0!}$$

$$0 \le |T_2(x) - f(x)| \le \frac{44}{3}e^4$$

7. **[13 points]** For this problem, consider the function $f(x) = (1+x)\sin(x)$.

(a) Write $T_2(x)$, the second Taylor polynomial for f with base $b = 0$.

$$f(x) = (1+x)\sin(x) \Rightarrow f(0) = (1+0)\sin(0) = 0$$

$$f'(x) = (1)\sin(x) + (1+x)\cos(x) \Rightarrow f'(0) = 0 + (1)(1) = 1$$

$$\left.\begin{aligned} f''(x) &= \cos(x) + \cos(x) - (1+x)\sin(x) \\ &= 2\cos(x) - (1+x)\sin(x) \end{aligned}\right\} \Rightarrow f''(0) = 2 - 0 = 2$$

$$T_2(x) = \underbrace{0} + \frac{1}{1!}\underbrace{1}(x-0) + \frac{1}{2!}\underbrace{2}(x-0)^2$$

$$\boxed{T_2(x) = x + x^2}$$

$$(1+x)\sin(x) \approx x + x^2$$

(b) Find (and justify) an error bound for $|f(x) - T_2(x)|$ on the interval $[-0.01, 0.01]$.

$$\begin{aligned} f'''(x) &= -2\sin(x) - [(1)\sin(x) + (1+x)\cos(x)] \\ &= -3\sin(x) - (1+x)\cos(x) \end{aligned}$$

$$|f'''(x)| = |3\underbrace{\sin(x)}_{\leq 1} + (1+x)\underbrace{\cos(x)}_{\leq 1}| \leq |3 + (1+x)| \leq 4.01$$

NOT A "TIGHT" BOUND, BUT A TIGHT BOUND IS NOT REQUIRED!

SO

$$|f(x) - T_2(x)| \leq \frac{4.01}{3!}|x-0|^3 \leq \boxed{\frac{4.01}{3!}(0.01)^3}$$

7. (12 pts) Let $f(x) = 1 + x + x^2 + 3x^3$.

(a) Find the second-degree Taylor polynomial, $T_2(x)$, for $f(x)$ based at $b = 1$.

$f(x) = 1 + x + x^2 + 3x^3 \Rightarrow f(1) = 1 + (1) + (1)^2 + 3(1)^3 = 6$

$f'(x) = 0 + 1 + 2x + 9x^2 \Rightarrow f'(1) = 1 + 2(1) + 9(1)^2 = 12$

$f''(x) = 0 + 2 + 18x \Rightarrow f''(1) = 2 + 18(1) = 20$

$T_2(x) = \underbrace{6} + \frac{1}{1!}\underbrace{12}(x-1) + \frac{1}{2!}\underbrace{20}(x-1)^2$

$\boxed{T_2(x) = 6 + 12(x-1) + 10(x-1)^2}$

(b) Determine an interval around $b = 1$ on which

$$|T_2(x) - f(x)| < 0.024.$$

CONSIDER $[1-a, 1+a]$. WE WILL SOLVE FOR a.

$|f'''(x)| = 18 \leq 18$

$|T_2(x) - f(x)| \leq \frac{18}{3!}|x-1|^3 \leq \underbrace{\frac{18}{6}|1+a-1|^3}_{3a^3}$

WANT $3a^3 < 0.024$

$\Rightarrow a^3 < 0.008$

$\Rightarrow a < 0.2$

$(1-0.2, 1+0.2) = \boxed{(0.8, 1.2)}$

F'17 Final Problem 7 – Lecture C

Let $f(x) = \sqrt{x^3}$. $= x^{3/2}$

(a) Find the second Taylor Polynomial based at b = 1.

(b) Find an upper bound for $|T_2(x) - f(x)|$ on the interval $[1-a, 1+a]$. $0 \le a < 1$

(c) Find a value of a between 0 and 1 such that

$$|T_2(x) - f(x)| \le 0.004$$

for all x in $[1-a, 1+a]$.

(a) $f'(x) = \frac{3}{2} x^{1/2} = \frac{3}{2}\sqrt{x}$

$f''(x) = \frac{3}{4} x^{-1/2} = \frac{3}{4\sqrt{x}}$

$$T_2(x) = \underbrace{1}_{f(1)} + \underbrace{\tfrac{3}{2}}_{f'(1)}(x-1) + \frac{1}{2!}\underbrace{\tfrac{3}{4}}_{f''(1)}(x-1)^2$$

$$\boxed{x^{3/2} \approx 1 + \tfrac{3}{2}(x-1) + \tfrac{3}{8}(x-1)^2}$$

(b) $\boxed{\text{STEP 1}}$ $f'''(x) = -\frac{3}{8} x^{-3/2} = \frac{-3}{8x^{3/2}}$

$|f'''(x)| = \frac{3}{8}\frac{1}{x^{3/2}} \le \frac{3}{8}\frac{1}{(1-a)^{3/2}}$

LARGEST WHEN DENOM. SMALLEST

$\boxed{\text{STEP 2}}$

ERROR $= |T_2(x) - f(x)| \le \frac{\quad}{3!}|x-1|^3$

1+a or 1−a

$$= \boxed{\frac{1}{3!}\frac{3}{8}\frac{1}{(1-a)^{3/2}}a^3} = \frac{1}{16}\frac{a^3}{(1-a)^{3/2}}$$

(c) WANT

$\frac{1}{16}\left(\frac{a^3}{\sqrt{1-a}^{\,3}}\right) = 0.004$

$\Rightarrow \frac{a^3}{(\sqrt{1-a})^3} = 0.064$

$\Rightarrow \frac{a}{\sqrt{1-a}} = (0.064)^{1/3} = 0.4$

$a = 0.4\sqrt{1-a}$

$a^2 = 0.16(1-a)$

$a^2 + 0.16a - 0.16 = 0$

$a = \frac{-0.16 \pm \sqrt{0.16^2 - 4(1)(-0.16)}}{2(1)}$

$a \approx -0.48792$

0.32792

Sp'17 Final Problem 9 – Both Lectures

Let $T_n(x)$ be the nth Taylor polynomial for $f(x) = e^{3x}$ based at $b = 0$. Find any value of n such that

$$|f(x) - T_n(x)| \le 0.01$$

for all x in $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

$f(x) = e^{3x}$
$f'(x) = 3e^{3x}$
$f''(x) = 3^2 e^{3x}$
$f'''(x) = 3^3 e^{3x}$

$f^{(k)}(x) = 3^k e^{3x}$

$T_n(x) = \underline{1} + \underline{3}x + \frac{1}{2!}\underline{3^2}x^2 + \frac{1}{3!}\underline{3^3}x^3 + \cdots + \frac{1}{n!}\underline{3^n}x^n$ } BONUS QUESTION DID NOT ASK FOR THIS

$$\boxed{T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} 3^k x^k}$$

$n = 1 \Rightarrow |f''(x)| = 3^2 e^{3x} \le 3^2 e^{3(\frac{1}{3})} = 3^2 e \Rightarrow$ ERROR $\le \frac{3^2 e}{2!}|x|^2 \le \frac{3^2 e}{2}\left(\frac{1}{3}\right)^2 = \frac{e}{2}$

$n = 2 \Rightarrow |f'''(x)| = 3^3 e^{3x} \le 3^3 e \Rightarrow$ ERROR $\le \frac{3^3 e}{3!}\left|\frac{1}{3}\right|^3 = \frac{e}{3!}$

$n = 3 \Rightarrow |f^{(4)}(x)| = 3^4 e^{3x} \le \cancel{3}^4 e \Rightarrow$ ERROR $\le \frac{3^4 e}{4!}\left|\frac{1}{3}\right|^4 = \frac{e}{4!}$

PATTERN: FOR $T_n(x)$ WE GET ERROR $\le \frac{e}{(n+1)!}$

n=5 WORKS! OR ANY LARGER n!

n	1	2	3	4	5	6
$\frac{e}{(n+1)!}$	$\frac{e}{2} \approx 1.359$	$\frac{e}{3!} \approx 0.453$	$\frac{e}{4!} \approx 0.11$	$\frac{e}{5!} \approx 0.023$	$\frac{e}{6!} \approx 0.0038$	$\frac{e}{7!} = 0.000539$

9. (12 pts) For **ALL** parts below, consider Taylor polynomials for $g(x) = e^{x/2}$ based at $b = 1$.

(a) Find the third Taylor polynomial, $T_3(x)$, for $g(x)$ based at $b = 1$.

$f(x) = e^{x/2}$ $f(b) = \sqrt{e}$

$f'(x) = \frac{1}{2}e^{x/2}$ $f'(b) = \frac{1}{2}\sqrt{e}$

$f''(x) = \frac{1}{4}e^{x/2}$ $f''(b) = \frac{1}{4}\sqrt{e}$

$f'''(x) = \frac{1}{8}e^{x/2}$ $f'''(b) = \frac{1}{8}\sqrt{e}$

$$\boxed{T_3(x) = \sqrt{e} + \frac{1}{2}\sqrt{e}(x-1) + \frac{1}{8}\sqrt{e}(x-1)^2 + \frac{1}{48}\sqrt{e}(x-1)^3}$$

(b) On the interval $I = [0, 2]$, for which of the values of n below does Taylor's inequality guarantee that $|f(x) - T_n(x)| < 0.001$?
You **must** show enough error bound calculations to justify your answer.

Circle **ALL** that apply: $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$

$$f^{(n)}(x) = \frac{1}{2^n}e^{x/2}$$

On $[0,2]$ this is at most $\frac{1}{2^n}e$.

So: $$|f(x) - T_n(x)| < \left(\frac{1}{(n+1)!}\right)\underbrace{\left(\frac{1}{2^{n+1}}e\right)}_{M} 1^{n+1} = \frac{e}{2^{n+1}(n+1)!}$$

$n=2$: $\frac{e}{2^3 \cdot 3!} \approx .057$ no.

$n=3$: $\frac{e}{2^4 \cdot 4!} \approx .007$ no.

$n=4$: $\frac{e}{2^5 \cdot 5!} \approx .0007$ yeah!