TEST PREP on 15.1/15.2 - Dr. Loveless

Ch. 15 Problem Solving Template - Here is my suggested template for how to approach ANY problem in chapter 15. You will come to better understand this over the coming days.

- *STEP 1*: Integrand? Find height $z = ??$. Then write $\iint_D ??\, dA$
- *STEP 2*: Draw the region, D, on the xy-plane.
  - Take any boundary condition involving only x and/or y and draw it on the xy-plane.
  - Draw any curves that correspond to when the surfaces (*i.e.* z's) intersect. (more on this later, not needed for this problem)
- *STEP 3*: Set up bounds... we will be discussing your options for doing this in 15.1-15.3.
- *STEP 4*: Evaluate... this is where you get to review integration.

Okay, try this...

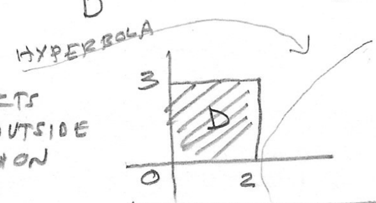
Spring 2011 - Exam 2 - Dr. Loveless (a rectangular region)

3(a) Set up and evaluate a double integral to find the volume of the solid that is below the surface $z + 3x^2 - 5y^2 = 12$ and bounded by the planes, $x = 0$, $x = 2$, $y = 0$, $y = 3$, and $z = 0$.
(Hint: Try to use the template and ask your TA for help, make sure you get to the evaluation step and practice evaluating).

$\boxed{\text{STEP 1}}$ $z = 12 - 3x^2 + 5y^2$ & $z = 0$ $\Rightarrow \iint_D 12 - 3x^2 + 5y^2\, dA$

$\boxed{\text{STEP 2}}$ - GIVEN $x = 0, x = 2, y = 0, y = 3$

- ASIDE: $\left[\begin{array}{l} 12 - 3x^2 + 5y^2 = 0 \\ \Rightarrow 12 = 3x^2 - 5y^2 \end{array}\right]$ INTERSECTS GROUND OUTSIDE OF REGION



$\boxed{\text{STEP 3}}$ $\int_0^2 \int_0^3 12 - 3x^2 + 5y^2\, dy\, dx$ OR $\int_0^3 \int_0^2 12 - 3x^2 + 5y^2\, dx\, dy$

$\boxed{\text{STEP 4}}$ $\int_0^2 12y - 3x^2 y + \frac{5}{3}y^3 \Big|_0^3 dx$ OR $\int_0^3 12x - x^3 + 5y^2 x \Big|_0^2 dy$

$= \int_0^2 36 - 9x^2 + 45\, dx$

$= 81x - 3x^3 \Big|_0^2$

$= 162 - 24$

$= \boxed{138}$

$= \int_0^3 24 - 8 + 10y^2\, dy$

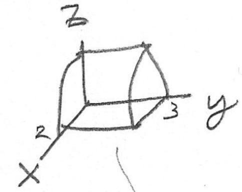
$= 16y + \frac{10}{3}y^3 \Big|_0^3$

$= 48 + 90$

$= \boxed{138}$

Spring 2019 - Exam 2 - Dr. Loveless (another rectangular region)

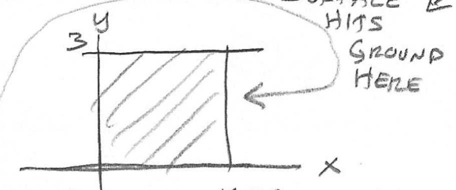
3(a) Find the volume of the solid in the $\underbrace{\textit{first octant}}_{x \geq 0,\ y \geq 0,\ z \geq 0}$ bounded by the parabolic cylinder $z = 12 - 3x^2$ and the plane $y = 3$.



Here is the template again, can you fill it in?

- *STEP 1*: Integrand? $\iint_D ??\, dA$ $= \iint_D 12 - 3x^2\, dA$
- *STEP 2*: Draw the region on the xy-plane.
  - Draw given x and/or y boundaries. ← $x = 0, y = 0, y = 3$
  - Draw any curves that correspond to surfaces (*i.e.* z's) intersecting. You need this now!
- *STEP 3*: Bounds.
- *STEP 4*: Evaluate.

$\left.\begin{array}{l} z = 12 - 3x^2 \\ z = 0 \end{array}\right\}$ $12 - 3x^2 = 0$, $x^2 = 4$, $x = \pm 2$, $\boxed{x = 2}$



$$\int_0^2 \int_0^3 12 - 3x^2\, dy\, dx = \int_0^3 \int_0^2 12 - 3x^2\, dx\, dy$$

$$= \int_0^2 12y - 3x^2 y \Big|_0^3 dx$$
$$= \int_0^2 36 - 9x^2\, dx$$
$$= 36x - 3x^3 \Big|_0^2 = 72 - 24$$
$$= \boxed{48}$$

OR

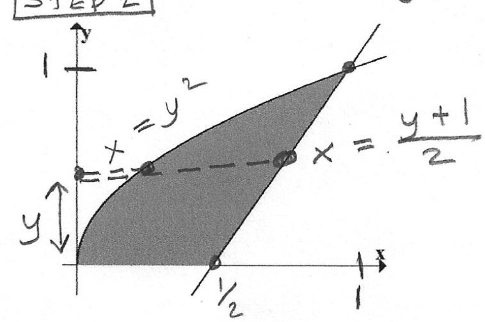
$$= \int_0^3 12x - x^3 \Big|_0^2 dy$$
$$= \int_0^3 24 - 8\, dy$$
$$= 16y \Big|_0^3$$
$$= \boxed{48}$$

Winter 2016 - Exam 2 - Dr. Loveless (a general region).

3(b) Let D be the region in the first quadrant of the xy-plane bounded by $y = 2x - 1$ and $y^2 = x$ (as shown). Evaluate $\iint_D 4x\, dA$. ← $\boxed{\text{STEP 1}}$

MUCH EASIER TO USE y!

$\boxed{\text{STEP 2}}$



FOR EACH y WITH $0 \leq y \leq 1$

WE HAVE LEFT $= y^2 \leq x \leq \frac{y+1}{2}$ = RIGHT

$\boxed{\text{STEP3}}$ $$\int_0^1 \int_{y^2}^{\frac{y+1}{2}} 4x\, dx\, dy$$

$\boxed{\text{STEP4}}$ $$= \int_0^1 2x^2 \Big|_{y^2}^{\frac{y+1}{2}} dy$$

$$= 2\frac{(y+1)^2}{4} - 2y^4 \Big|_0^1$$
$$= \frac{1}{2}\frac{1}{3}(y+1)^3 - \frac{2}{5}y^5 \Big|_0^1$$
$$= \left(\frac{8}{2 \cdot 3} - \frac{2}{5}\right) - \left(\frac{1}{6} - 0\right)$$
$$= \frac{4}{3} - \frac{2}{5} - \frac{1}{6} = \frac{40}{30} - \frac{12}{30} - \frac{5}{30}$$
$$= \boxed{\frac{23}{30}}$$

OR

HARDER WAY

$$\underbrace{\int_0^{\frac{1}{2}} \int_0^{\sqrt{x}} 4x\, dy\, dx}_{= \sqrt{2}/5} + \underbrace{\int_{\frac{1}{2}}^1 \int_{2x-1}^{\sqrt{x}} 4x\, dy\, dx}_{\left(\frac{23}{30} - \frac{\sqrt{2}}{5}\right)}$$

$$= \boxed{23/30}$$

A Few Comments

Now that you have seen a few problems, you may appreciate it better when I say there are two main things students tend to struggle with in chapter 15.1 and 15.2:

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1. **Working with regions** - This is a big skill you are practicing.
   - **Always**, draw the region and decide whether you want to use x or y.
     - same top/bottom boundary sides throughout region... use x.
     - same left/right boundary sides throughout region... use y.
   - Label x, or y, on the appropriate axis and label all boundaries in terms of that variable.

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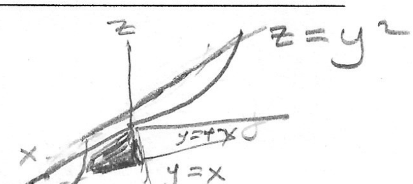
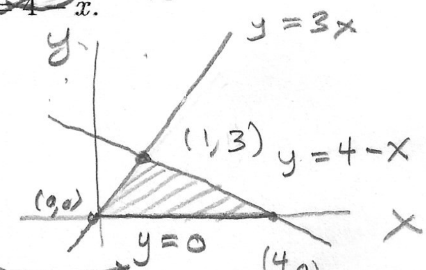
2. **Integration Review** - You WILL have to use your integration in chapter 15 including:
   - Simplification, Substitution (reversing chain rule), By Parts (products and logs) and Trig (integrating powers of Sine and Cosine).
   - Working with bounds of integration (how to change if you do substitution, evaluating, etc).

All the things above were essential skills in Calculus 2. See my website and lecture videos for review.



Here is another one...
Spring 2014 - Exam 2 - Dr. Loveless (follow my template!)

3(b) Find the volume of the solid bounded by $z = y^2$, $z = 0$, $y = 3x$ and $y = 4 - x$.



$\boxed{\text{STEP 1}}$ $\iint_D y^2\,dA$

$\boxed{\text{STEP 2}}$ - GIVEN $y = 3x$, $y = 4 - x$

- z's INTERSECT?

$z = y^2$ & $z = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0$

$\boxed{\text{STEP 3}}$ USE y!

$$\int_0^3 \int_{y/3}^{4-y} y^2\,dx\,dy$$

$\boxed{\text{STEP 4}}$

$$\int_0^3 y^2 x \Big|_{y/3}^{4-y} dy \quad \leftarrow -\tfrac{4}{3}y^3$$

$$= \int_0^3 4y^2 - y^3 - \tfrac{1}{3}y^3\,dy$$

$$= \tfrac{4}{3}y^3 - \tfrac{1}{3}y^4 \Big|_0^3 = 36 - 27$$

$$= \boxed{9}$$

OR

$$\underbrace{\int_0^1 \int_0^{3x} y^2\,dy\,dx}_{} + \underbrace{\int_1^4 \int_0^{4-x} y^2\,dy\,dx}_{}$$

$$= \tfrac{9}{4} + \tfrac{27}{4}$$

$$= \tfrac{36}{4} = \boxed{9}$$

Winter 2016 - Exam 2 - Dr. Loveless (draw the region first!) $\leftarrow$ GIVEN $\boxed{\text{STEP 1}}$ ✓

4(a) Reverse the order of integration and evaluate $\displaystyle\int_0^2 \int_{2y}^4 8\sqrt{x^2+1}\, dx\, dy.$

DRAW FIRST!!!

$x = 4 \leftarrow$ "RIGHT"

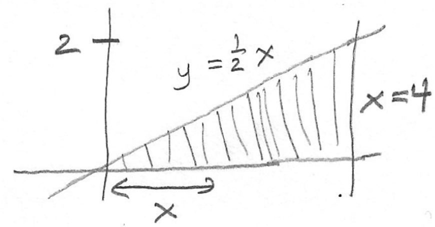
$x = 2y \leftarrow$ "LEFT"

$\boxed{\text{STEP 2}}$ GIVEN $\quad 0 \le y \le 2 \Rightarrow 2y \le x \le 4 \Rightarrow$

$\boxed{\text{STEP 3}}$

$$\int_0^4 \int_0^{\frac{1}{2}x} 8\sqrt{x^2+1}\, dy\, dx$$

TOP: $\frac{1}{2}x$ BOT: 0



$\boxed{\text{STEP 4}}$

$$\int_0^4 8\sqrt{x^2+1}\; y \Big|_0^{\frac{1}{2}x} dx$$

$$= \int_0^4 4\sqrt{x^2+1}\; x\, dx$$

$u = x^2+1$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$= \int_1^{17} 2\sqrt{u}\, du$$

$$= \frac{4}{3} u^{3/2} \Big|_1^{17} = \boxed{\frac{4}{3}(17)^{4/3} - 4/3}$$

Spring 2014 - Exam 2 - Dr. Loveless (draw the region) $\leftarrow$ GIVEN / $\boxed{\text{STEP 1}}$ ✓

3(a) Reverse the order of integration and evaluate $\displaystyle\int_0^3 \int_{x^2}^9 x e^{y^2}\, dy dx.$ $\leftarrow$ DRAW FIRST

$y = 9 \leftarrow$ "TOP"

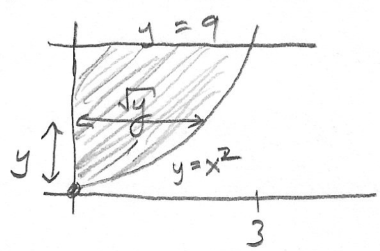
$y = x^2 \leftarrow$ "BOT"

$\boxed{\text{STEP 2}}$ GIVEN $\quad 0 \le x \le 3, \quad x^2 \le y \le 9 \Rightarrow$

$\boxed{\text{STEP 3}}$ RIGHT

$$\int_0^9 \int_0^{\sqrt{y}} x e^{y^2}\, dx\, dy$$

$0 \leftarrow$ LEFT



$\boxed{\text{STEP 4}}$ $\displaystyle\int_0^9 \frac{1}{2} x^2 e^{y^2} \Big|_0^{\sqrt{y}} dy$

$$\int_0^9 \frac{1}{2} y e^{y^2}\, dy$$

$u = y^2$

$du = 2y\, dy$

$\frac{1}{2} du = y\, dy$

$$= \int_0^{81} \frac{1}{4} e^u\, du$$

$$= \frac{1}{4} e^u \Big|_0^{81} = \boxed{\frac{1}{4} e^{81} - \frac{1}{4}}$$

NOTE: For a problem describing a solid between TWO different surfaces (other than $z = 0$), then I have a slightly amended step 1 to the template:

- *STEP 1*: Integrand(s)? If there are two surfaces giving height (two z's), set up TWO double integrals, one for each z, and then subtract to find the volume between (bigger minus smaller).
- *STEP 2*: Draw the region.
  - Draw given x and/or y boundaries.
  - Draw any curves that correspond to surfaces (*i.e.* z's) intersecting.
- *STEP 3*: Bounds.
- *STEP 4*: Evaluate.

Here is an example where this happens...

Spring 2013 Honors - Exam 2 - Dr. Loveless (use my template)

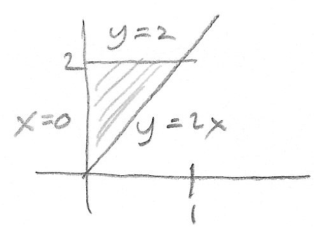
2(a) Set up and evaluate a double integral to find the volume of the solid that is below the surface $z - 3x^2y - 1 = 0$, above the surface $z = 1$ and between the planes $x = 0$, $y = 2$, and $y = 2x$.

Hints:

STEP 1

- The solid is between $z = 1 + 3x^2y$ and $z = 1$, so we write $\iint_D 1 + 3x^2y\,dA$ and $\iint_D 1\,dA$.
- The final answer will be $\iint_D 1 + 3x^2y\,dA - \iint_D 1\,dA = \iint_D 3x^2y\,dA$.

STEP 2 GIVEN $x=0, y=2, y=2x$ $\longrightarrow$



STEP 3

$$\int_0^1 \int_{2x}^2 3x^2y\,dy\,dx \quad \text{OR} \quad = \int_0^2 \int_0^{y/2} 3x^2y\,dx\,dy$$

$$= \int_0^1 \frac{3}{2}x^2y^2 \Big|_{2x}^2 dx$$

$$= \int_0^1 6x^2 - 6x^4\,dx$$

$$= 2x^3 - \frac{6}{5}x^5 \Big|_0^1$$

$$= 2 - \frac{6}{5} = \boxed{4/5}$$

$$= \int_0^2 x^3y \Big|_0^{y/2} dy$$

$$= \int_0^2 \frac{1}{8}y^4\,dy$$

$$= \frac{1}{40}y^5 \Big|_0^2$$

$$= \frac{32}{40} = \boxed{4/5}$$