**TEST PREP on 13.1 and 13.2 - Dr. Loveless**

***Test Prep Reminder***: These problems mostly come *directly* from the Dr. Loveless old exam archive. You can find solutions in that archive after class. Keep asking yourself, could I really do this on a test? How can I be more efficient? And how can I check my answers?

***Quiz note***: You should expect at least one of these first problems to appear on the quiz at the end of this week (perhaps with numbers slightly changed), so there is an added reason to take good notes on these problems and ask your TA and classmates lots of questions.

***13.1-13.2 Extra Help***: Here are a few of my review materials on this topic if you need extra help:

- Section 13.1 Review - includes additional examples on visualizing, parameterizing an intersection, finding an intersection of curves, and what it means for two object in motion to collide.
- Section 13.2 Review - a basic fact sheet on tangent vectors, tangent lines and notation.
- Calculus Fact Sheet - Derivatives and integrals you can quote. You will have to do some derivatives in chapter 13 (and a few basic integrals... know substitution, by parts, and how to find "+C").

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From Fall 2018, Exam 1, Loveless (quick "surface of motion" question)

1(a) Which of the following functions describe points that are always on the curve of intersection of the surfaces $x^2 + z^2 = 4$ and $x = 2y$: (select ALL for which every point on the curve is on the intersection).

ii. Which of the following vector functions give points that are always on the curve of intersection of $x^2 + z^2 = 4$ and $x = 2y$:

$x = 2y$ ✓ $x^2 + z^2 = 4$ ✓
(i) $\langle t, \frac{1}{2}t, \sqrt{4 - t^2} \rangle$

$x = 2y$ ✓, $x^2 + z^2 = 4$ ✓
(ii) $\langle 2\cos(t), \cos(t), 2\sin(t) \rangle$

(iii) $\langle 2\sin(t^3), \sin(t^3), 2\cos(t^3) \rangle$
$x = 2y$ ✓ $x^2 + z^2 = 4$ ✓

(iv) $\langle 2t, t, 0 \rangle$
$x = 2y$ ✓ $x^2 + z^2 = 4t^2 \neq 4$

From Spring 2013 (Honors), Exam 1, Loveless (finding a tangent line... this was half a page).

4(a) Dr. Loveless has motion sickness. You trick him into getting on a roller coaster that follows the path given by the vector function: $\mathbf{r}(u) = \langle 20\sin(u), 24u, 20\cos(u) + 40 \rangle$. When the ride gets to the point $(x, y, z) = (10\sqrt{3}, 8\pi, 50)$, Dr. Loveless' calculator falls out of his pocket. Assume the calculator follows the path of the tangent line (we will assume there is no gravity, ha).

If the $xy$-plane is the ground, at what location $(x, y, z)$ does the calculator land on the ground?

$20\sin(u) = 10\sqrt{3}$, $24u = 8\pi$, $20\cos(u) + 40 = 50 \Rightarrow u = \pi/3$

$\vec{r}\,'(u) = \langle 20\cos(u), 24, -20\sin(u) \rangle$

$\vec{r}\,'(\pi/3) = \langle 10, 24, -10\sqrt{3} \rangle$

TANGENT LINE:
$x = 10\sqrt{3} + 10t$
$y = 8\pi + 24t$
$z = 50 - 10\sqrt{3}t$

$z = 0 \Rightarrow t = \frac{50}{10\sqrt{3}} = \frac{5}{\sqrt{3}} = \frac{5}{3}\sqrt{3}$

$$\boxed{\begin{aligned}(x, y, z) &= \left(10\sqrt{3} + \tfrac{50}{3}\sqrt{3},\ 8\pi + 24\tfrac{5}{3}\sqrt{3},\ 0\right)\\ &= \left(\tfrac{80}{3}\sqrt{3},\ 8\pi + 40\sqrt{3},\ 0\right)\end{aligned}}$$

From Winter 2016, Exam 1, Loveless (finding a tangent line as well as intersecting curves).

4. At time $t = 0$, a egg is set in motion toward Dr. Loveless by a disgruntled student (Dr. Loveless is sitting on the $xy$- plane at the point where the egg will eventually hit him). The egg's path is described parametrically by $x = t$, $y = 8\sqrt{t+1}$, $z = 15t - t^2$.

   (a) (3 pts) At the time $t = 0$, find a vector that is tangent to the curve and has length 5.

   $\vec{r}_1(t)$

   $\vec{r}_1\,'(t) = \langle 1, \frac{4}{\sqrt{t+1}}, 15-2t \rangle$, $\vec{r}_1\,'(0) = \langle 1, 4, 15 \rangle$

   $|\vec{r}_1\,'(0)| = \sqrt{1+16+225} = \sqrt{242} = 11\sqrt{2}$

   + or − okay

   $\boxed{\frac{5}{\sqrt{242}} \langle 1, 4, 15 \rangle = \langle \frac{5}{\sqrt{242}}, \frac{20}{\sqrt{242}}, \frac{75}{\sqrt{242}} \rangle}$

   (b) Find parametric equations for the tangent line at the positive time when the egg hits the $xy$-plane.

   HITS $xy$-PLANE $\Leftrightarrow$ $z = 15t - t^2 \stackrel{?}{=} 0 \Leftrightarrow t(15-t) = 0$

   $t = 0$ or $t = 15$

   $\vec{r}_1(15) = \langle 15, 8\sqrt{16}, 0 \rangle = \langle 15, 32, 0 \rangle$

   $\vec{r}_1\,'(15) = \langle 1, \frac{4}{\sqrt{16}}, 15 - 2(15) \rangle = \langle 1, 1, -15 \rangle$

   $$\boxed{\begin{aligned} x &= 15 + t \\ y &= 32 + t \\ z &= 0 - 15t \end{aligned}}$$

   (c) A rock is also flying through the air following the path $\overbrace{x = 3,\ y = 14 + u,\ z = 28 + u^3}^{\vec{r}_2(u)}$. The path of the rock and the path of the egg intersect (unfortunately for Dr. Loveless, the rock and egg don't collide). Find the (acute) angle of intersection of the two paths. Give your final answer in degrees rounded to two digits after the decimal.

   INTERSECTION: $t \stackrel{?}{=} x \stackrel{?}{=} 3$ $\quad t = 3$

   $8\sqrt{t+1} \stackrel{?}{=} y \stackrel{?}{=} 14 + u \Rightarrow 16 = 14 + u \Rightarrow u = 2$

   $15t - t^2 \stackrel{?}{=} z \stackrel{?}{=} 28 + u^3 \Rightarrow$ CHECK $\underbrace{15(3) - (3)^2}_{36} \stackrel{?}{=} \underbrace{28 + (2)^3}_{36}$ ✓

   $\vec{r}_1\,'(3) = \langle 1, 2, 9 \rangle = \vec{u}$

   $\vec{r}_2\,'(u) = \langle 0, 1, 3u^2 \rangle$ $\quad \vec{r}_2\,'(2) = \langle 0, 1, 12 \rangle = \vec{v}$

   $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{0 + 2 + 108}{\sqrt{1+4+81}\sqrt{0+1+144}} = \frac{110}{\sqrt{86}\sqrt{145}}$

   $\boxed{\theta = \cos^{-1}\left(\frac{110}{\sqrt{86}\sqrt{145}}\right) \approx 9.92 \text{ degrees}}$

From Spring 2014, Exam 1, Loveless (finding a tangent line as well as intersecting a curve and a surface).

4. You are sitting at the origin on the surface $4z - x^2 - y^2 = 0$. You launch a water balloon into the air and its position at time $t$ seconds is given roughly by the vector function $\mathbf{r}(t) = \langle t, 2t, 20t - 5t^2 \rangle$.

**(a) Give the two word name of this surface.**

CIRCULAR PARABOLOID (ELLIPTICAL PARABOLOID, OKAY)

**(b) Your math instructor just happens to be sitting at the location where the water balloon lands on the surface. Find the $(x, y, z)$ location where your math instructors is sitting.**

INTERSECTION: $4(20t - 5t^2) - t^2 - (2t)^2 = 0$

$80t - 20t^2 - t^2 - 4t^2 = 0$

$80t - 25t^2 = 0$

$5t(16 - 5t) = 0 \qquad t = 0$ or $\boxed{t = \frac{16}{5} = 3.2}$

$= 3.2 \Rightarrow \boxed{(x,y,z) = \left(\frac{16}{5}, \frac{32}{5}, \frac{64}{5}\right) = (3.2, 6.4, 12.8)}$

**(c) Find parametric equations for the tangent line to the path at $t = 2$.**

$\vec{r}(2) = \langle 2, 4, 40 - 20 \rangle = \langle 2, 4, 20 \rangle$

$\vec{r}\,'(t) = \langle 1, 2, 20 - 10t \rangle \qquad \vec{r}\,'(2) = \langle 1, 2, 0 \rangle$

$$\boxed{\begin{aligned} x &= 2 + t \\ y &= 4 + 2t \\ z &= 20 \end{aligned}}$$

ASIDE: THIS IS THE HIGHEST THE BALLOON GETS. THE TANGENT IS PARALLEL TO THE xy-PLANE.

**(d) Find the curvature at time $t = 2$.**

$\vec{r}\,'(2) = \langle 1, 2, 0 \rangle$

$\vec{r}\,''(t) = \langle 0, 0, -10 \rangle \Rightarrow \vec{r}\,''(2) = \langle 0, 0, -10 \rangle$

$$\vec{r}\,'(2) \times \vec{r}\,''(2) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 0 & -10 \end{vmatrix} = (-20 - 0)\vec{i} - (-10 - 0)\vec{j} + (0 - 0)\vec{k} = \langle -20, 10, 0 \rangle$$

$$K(2) = \frac{|\vec{r}\,'(2) \times \vec{r}\,''(2)|}{|\vec{r}\,'(2)|^3} = \frac{\sqrt{20^2 + 10^2 + 0^2}}{(1^2 + 2^2 + 0^2)^{3/2}} = \frac{\sqrt{500}}{5^{3/2}} = \frac{10\sqrt{5}}{5^{3/2}} = \frac{10}{5} = 2$$

ASIDE: THIS IS THE MAXIMUM CURVATURE FOR THIS CURVE