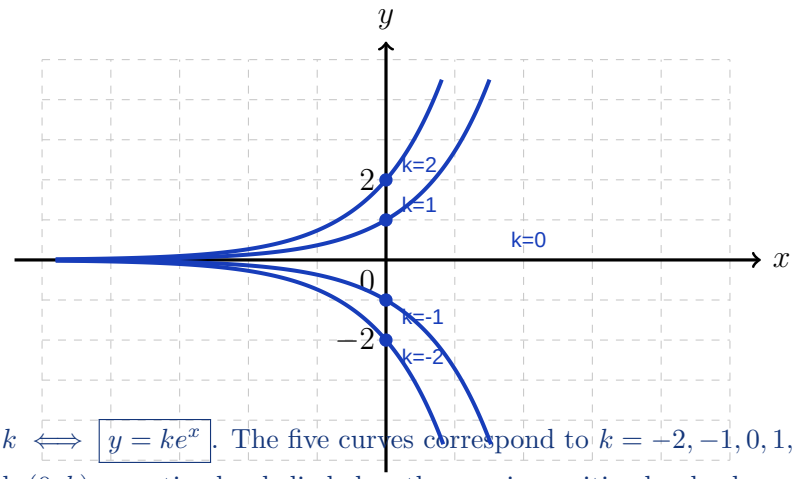
1. (4 pts) Level curves – *Section: 14.1*

Modeled on: WebAssign HW 14.1 contour and level-curve questions.

For $z = ye^{-x}$, sketch and label the level curves $z = k$ for $k = -2, -1, 0, 1, 2$ on the axes below. Show generally correct shapes; label each curve and its y-intercept.



Solution. $ye^{-x} = k \iff \boxed{y = ke^{x}}$. The five curves correspond to $k = -2, -1, 0, 1, 2$.
Each passes through $(0, k)$; negative levels lie below the x-axis, positive levels above it, and $k = 0$ is $y = 0$.

2. (8 pts) Tangent plane from two curves – *Sections: 14.3, 14.4*

Source: WebAssign HW 14.4, Problem 7 (Stewart 14.4.042).

A smooth surface S contains both curves $\mathbf{r}_1(t) = \langle 4 + 2t, 1 - t^2, 3 - 5t + t^2 \rangle$ and $\mathbf{r}_2(u) = \langle 3 + u^2, 2u^3 - 1, 2u + 1 \rangle$. Find its tangent plane at $P = (4, 1, 3)$. Explain how the curves determine it.

Solution. At $t = 0$ and $u = 1$ both curves pass through $P = (4, 1, 3)$.
$\mathbf{r}_1'(0) = \langle 2, 0, -5 \rangle$, $\mathbf{r}_2'(1) = \langle 2, 6, 2 \rangle$.
A normal vector is their cross product: $\langle 30, -14, 12 \rangle = 2\langle 15, -7, 6 \rangle$.
Thus $\boxed{15(x - 4) - 7(y - 1) + 6(z - 3) = 0}$, or $\boxed{15x - 7y + 6z = 71}$.

3. (4 pts) Implicit partial derivative – *Sections: 14.3, 14.5*

Source: Loveless, Spring 2024 Math 126 Exam 2, Problem 1(a). Similar practice: HW 14.5 implicit derivatives; Week 6 Thursday TP.

If $2\sin(\pi z) = \ln y + x^2yz - 4y$ defines $z = z(x, y)$, find $\partial z/\partial x$ at $(x, y, z) = (2, 1, 1)$.

$$2\pi\cos(\pi z)\frac{\partial z}{\partial x} = 2xyz + x^2y\frac{\partial z}{\partial x}$$

$$@(2,1,1) \quad -2\pi\frac{\partial z}{\partial x} = 4 + 4\frac{\partial z}{\partial x}$$

$$-4 = (4+2\pi)\frac{\partial z}{\partial x}$$

$$\frac{-4}{4+2\pi} = \frac{\partial z}{\partial x}$$

4. (4 pts) Gradient and steepest increase – *Section: 14.6*

Source: WebAssign HW 14.6, Problem 7 (Stewart 14.6.029.MI).

For $f(x, y) = 7\sin(xy)$, find the maximum rate of increase at $(0, 3)$ and a unit vector in that direction.

Solution. $\nabla f = 7\cos(xy)\langle y, x\rangle$, so $\nabla f(0, 3) = \langle 21, 0\rangle$.
Maximum rate $\boxed{21}$ in unit direction $\boxed{\langle 1, 0\rangle}$.

5. (7 pts) Find and classify – *Section: 14.7*

Source: WebAssign HW 14.7, Problem 3 (Stewart 14.7.010).

Find and classify all critical points of $f(x, y) = 6 - x^4 + 2x^2 - y^2$ using the second derivative test.

Solution. $f_x = -4x^3 + 4x = 4x(1 - x^2)$, $f_y = -2y$; critical points $(0, 0)$, $(\pm 1, 0)$.
$f_{xx} = 4 - 12x^2$, $f_{yy} = -2$, $f_{xy} = 0$, so $D = -2(4 - 12x^2)$.
At $(0, 0)$: $D = -8 < 0$, **saddle**. At $(\pm 1, 0)$: $D = 16 > 0$, $f_{xx} = -8 < 0$, **local maxima** ($f = 7$).

6. (13 pts) Global extrema on a square – *Section: 14.7*

Source: WebAssign HW 14.7, Problem 6 (Stewart 14.7.035).

Find the absolute maximum and minimum values of $f(x,y) = x^2 + y^2 + x^2y + 9$ on $D = \{(x,y) : |x| \le 1, |y| \le 1\}$. Check the interior, all four edges, and corners.

Solution. $f = x^2(1+y) + y^2 + 9$. Interior: $f_x = 2x(1+y) = 0$, $f_y = x^2 + 2y = 0$ give $(0,0)$, $f = 9$.
Edges $y = 1$: $f = 2x^2 + 10$, range $[10, 12]$; $y = -1$: $f = 10$.
Edges $x = \pm 1$: $f = y^2 + y + 10 = (y + \frac{1}{2})^2 + \frac{39}{4}$, range $[39/4, 12]$.
Corners are already included among the edges. Comparing values,
$\boxed{\text{Absolute minimum 9 at } (0,0)}$,
$\boxed{\text{absolute maximum 12 at } (\pm 1, 1)}$.

7. (8 pts) Lagrange multipliers – *Section: 14.8*

Adapted from: WebAssign HW 14.8, Problems 2–4 (basic constrained extrema).

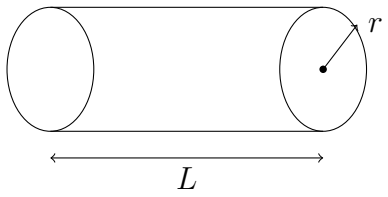
Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 2x + 3y$ subject to $x^2 + y^2 = 13$.

Solution. $\nabla f = \langle 2, 3\rangle = \lambda\langle 2x, 2y\rangle$, so $x = 1/\lambda$, $y = 3/(2\lambda)$.
The constraint $x^2 + y^2 = 13$ gives $\lambda = \pm\frac{1}{2}$. Thus $\boxed{(x,y) = (2,3),\ (-2,-3)}$.
$\boxed{\text{Maximum 13 at } (2,3)}$; $\boxed{\text{minimum } -13 \text{ at } (-2,-3)}$.

8. (12 pts) Cylindrical mailing tube – *Sections: 14.7, 14.8*

Source: WebAssign Optimization Applications HW, Problem 7 (Stewart 14.8.022).

A cylindrical mailing tube must have length plus girth at most 114 inches (girth is the circular end's circumference). Find the radius and length giving the greatest volume. Use substitution or Lagrange multipliers; identify the objective and constraint and explain why this is a maximum.



Solution. Length plus girth: $L + 2\pi r = 114$ at the maximum. The volume is $V = \pi r^2 L$.
Substituting $L = 114 - 2\pi r$ gives $V(r) = 114\pi r^2 - 2\pi^2 r^3$, $0 \le r \le 57/\pi$.
$V'(r) = 6\pi r(38 - \pi r)$, so the positive critical radius is $r = 38/\pi$.
Then $L = 114 - 2\pi(38/\pi) = 38$. Volume is 0 at either endpoint, so this is the global maximum.

$\boxed{r = 38/\pi \text{ inches}, \qquad L = 38 \text{ inches}}$.