1. (8 pts) Tangent plane – *Sections: 14.3, 14.4*

Source: Ostroff, Winter 2019 Exam 2, Problem 1. Similar practice: HW 14.4.

Write the equation of the tangent plane to $z = x^2y - x$ at $(2, 3, 10)$.

$\frac{\partial z}{\partial x} = 2xy - 1$, so at $(2,3,10)$, $\frac{\partial z}{\partial x} = 11$

$\frac{\partial z}{\partial y} = x^2$, so at $(2,3,10)$, $\frac{\partial z}{\partial y} = 4$

So the tangent plane is $z = 10 + 11(x-2) + 4(y-3)$

or $z = 11x + 4y - 24$

2. (11 pts) Find and classify – *Section: 14.7*

Source: Loveless, Spring 2023 Math 126 Exam 2, Problem 2(a). Similar practice: HW 14.7; Week 7 Tuesday Worksheet.

Find and classify all critical points of $f(x, y) = 6x^2 - 3x^2y + y^3 - 3y$. Show the second derivative test.

$f_x = 12x - 6xy \overset{?}{=} 0 \rightarrow 6x(2-y) = 0 \Rightarrow x = 0$ or $y = 2$

$f_y = -3x^2 + 3y^2 - 3 \overset{?}{=} 0$

$x = 0 \Rightarrow 3y^2 - 3 = 0 \Rightarrow y = \pm 1$

$y = 2 \Rightarrow -3x^2 + 12 - 3 = 0 \Rightarrow 3x^2 = 9 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

$f_{xx} = 12 - 6y$

$f_{yy} = 6y$

$f_{xy} = -6x$

$(0,1) \Rightarrow D = (6)(6) - (0)^2 = 12 > 0$ MIN

$(0,-1) \Rightarrow D = (18)(-6) - (1)^2 < 0$ SADDLE

$(-\sqrt{3}, 2) \Rightarrow D = (0)(12) - (6\sqrt{3})^2 < 0$ SADDLE

$(\sqrt{3}, 2) \Rightarrow D = (0)(12) - (6\sqrt{3})^2 < 0$ SADDLE

3. (15 pts) Felix walking on a surface – *Sections: 14.5, 14.6*

Source: Loveless, Autumn 2013 Math 324 Exam 2, Problem 1. Similar practice: HW 14.5 and 14.6; Week 7 Tuesday Worksheet.

Felix walks on $z = f(x,y) = \frac{1}{4}x^2 + \frac{1}{3}y^3 - \frac{1}{2}xy^2$ (all distances in miles). Positive y is north and positive x is east.

(a) **[5 pts]** Given $x(1) = 2$, $y(1) = 1$, $x'(1) = 3$, and $y'(1) = -4$, find dz/dt at $t = 1$ (hours). Give units.

CHAIN RULE!

$\frac{\partial z}{\partial x} = \frac{1}{2}x - \frac{1}{2}y^2 \quad \Rightarrow \quad \frac{dz}{dt} = \left(\frac{1}{2}x - \frac{1}{2}y^2\right)\frac{dx}{dt} + (y^2 - xy)\frac{dy}{dt}$

$\frac{\partial z}{\partial y} = y^2 - xy \qquad x=2, y=1, x'=3, y'=-4$

$\Rightarrow \frac{dz}{dt} = (1-\frac{1}{2})(3) + (1-2)(-4)$

$= \frac{3}{2} + 4 = \boxed{\frac{11}{2} \quad \frac{\text{miles}}{\text{hour}}}$

(b) **[5 pts]** At $(8,2)$, find the *unit* direction for steepest downhill motion.

STEEPEST UPHILL $\Leftrightarrow$ GRADIENT!

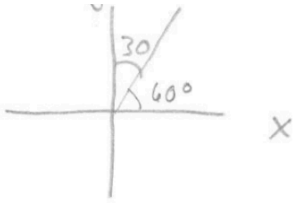
$\nabla f(8,2) = \langle \frac{1}{2}(8) - \frac{1}{2}(2)^2, (2)^2 - (8)(2)\rangle = \langle 4-2, 4-16\rangle = \langle 2, -12\rangle$

STEEPEST DOWNHILL $\Rightarrow$ $\langle -2, 12\rangle$ DIRECTION

UNIT VECTOR $\boxed{\frac{1}{\sqrt{4+144}}\langle -2,12\rangle = \frac{1}{\sqrt{148}}\langle -2,12\rangle = \frac{1}{\sqrt{37}}\langle -1,6\rangle}$

(c) **[5 pts]** At $(8,2)$, find the slope when Felix walks $30°$ east of north.

$\vec{u} = \langle \cos(60°), \sin(60°)\rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$ DIRECTION:



$D_{\vec{u}} f(8,2) = \nabla f(8,2) \cdot \vec{u}$

$= \langle 2, -12\rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$

$= \boxed{1 - 6\sqrt{3}}$

4. (14 pts) Global extrema on a half-disk – *Section: 14.7*

Source: Loveless, Winter 2024 Exam 2, Problem 4. Similar practice: HW 14.7; Week 7 Thursday Test Prep.

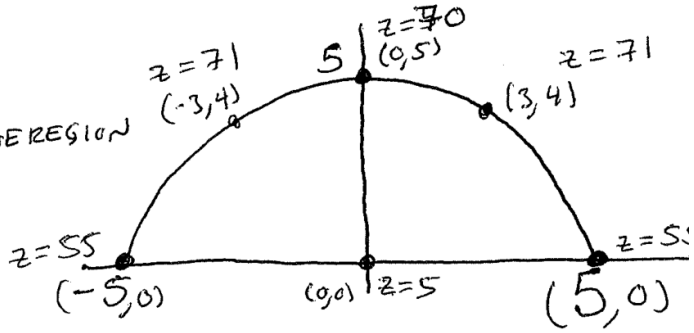
Find the absolute maximum and minimum values of $f(x,y) = 5 + 2x^2 + y^2 + 8y$ on $D = \{(x,y) : y \geq 0,\ x^2 + y^2 \leq 25\}$. Check interior points, every boundary component, and endpoints. State where the extrema occur.

above each boundary curve and that you checked all appropriate points and endpoints)

$f_x = 4x \stackrel{?}{=} 0 \to x = 0$

$f_y = 2y + 8 \stackrel{?}{=} 0 \to y = -4$

$(0,-4)$ IS NOT IN THE REGION



I $y = 0,\ -5 \leq x \leq 5 \Rightarrow z = f(x,0) = 5 + 2x^2$

$z' = 4x = 0 \Rightarrow x = 0$

$x = -5 \Rightarrow f(-5,0) = 55$

$x = 0 \Rightarrow f(0,0) = 5$

$x = 5 \Rightarrow f(5,0) = 55$

II $x^2 + y^2 = 25,\ \begin{matrix} 0 \leq y \leq 5 \\ -5 \leq x \leq 5 \end{matrix}$ TWO WAYS

$x^2 = 25 - y^2$

$z = 5 + 2(25 - y^2) + y^2 + 8y$

$z = 55 - 2y^2 + y^2 + 8y$

$z = 55 - y^2 + 8y$

$z' = -2y + 8 = 0 \Rightarrow y = 4$

$y = 0 \Rightarrow z = f(\pm 5, 0) = 55$

$y = 4 \Rightarrow z = f(\pm 3, 4) = 55 - 16 + 32 = 71$

$y = 5 \Rightarrow z = f(0,5) = 70$

$y = \sqrt{25 - x^2}$

$z = 5 + 2x^2 + 25 - x^2 + 8\sqrt{25 - x^2}$

$z' = 2x + \frac{-8x}{\sqrt{25-x^2}} \stackrel{?}{=} 0$

$2x\sqrt{25-x^2} - 8x = 0$

$2x(\sqrt{25-x^2} - 4) = 0$

$x = 0$ or $25 - x^2 = 16$

$x^2 = 9$

$x = \pm 3$

$x = -5 \to z = 55$

$x = -3 \to z = 71$

$x = 0 \to z = 70$

$x = 3 \to z = 71$

$x = 5 \to z = 55$

global max: $z =$ 71

global min: $z =$ 5

5. (12 pts) Aquarium material costs – *Sections: 14.7, 14.8*

Source: WebAssign Optimization Applications HW, Problem 5 (Stewart 14.7.054).

An open-top rectangular aquarium has slate on the bottom and glass on the four sides. Its volume is V, and slate costs three times as much per unit area as glass. Find the dimensions minimizing material cost, in terms of V. Clearly give the cost and volume constraint.

Let x, y be the base dimensions and h the height. Taking the glass cost per square unit as 1, the relative material cost and volume are

$$C = 3xy + 2xh + 2yh, \qquad xyh = V, \qquad x, y, h > 0.$$

Method 1: Substitution. From $h = V/(xy)$,

$$C(x, y) = 3xy + \frac{2V}{x} + \frac{2V}{y}, \quad C_x = 3y - \frac{2V}{x^2}, \quad C_y = 3x - \frac{2V}{y^2}.$$

Setting $C_x = C_y = 0$ gives $3x^2y = 3xy^2 = 2V$, so $x = y$ and $x^3 = 2V/3$. Thus

$$\boxed{x = y = \sqrt[3]{\frac{2V}{3}}, \qquad h = \frac{3}{2}\sqrt[3]{\frac{2V}{3}}.}$$

The cost increases toward infinity as a positive dimension approaches 0 (while V stays fixed), or as a base dimension grows without bound; this interior solution is the global minimum.

Method 2: Lagrange multipliers. Let $g(x, y, h) = xyh = V$; then $\nabla C = \lambda \nabla g$ gives

$$\begin{aligned} 3y + 2h &= \lambda yh, \\ 3x + 2h &= \lambda xh, \qquad xyh = V. \\ 2x + 2y &= \lambda xy. \end{aligned}$$

Multiplying the first two equations by x and y respectively, then subtracting, gives $2h(x-y) = 0$. Hence $x = y = a$. The third equation gives $\lambda = 4/a$; the first then gives $3a + 2h = 4h$, so $h = 3a/2$.

$$a^2\left(\frac{3a}{2}\right) = V \quad \Longrightarrow \quad \boxed{x = y = \sqrt[3]{\frac{2V}{3}}, \quad h = \frac{3}{2}\sqrt[3]{\frac{2V}{3}}.}$$