1. (6 pts) Partial derivatives – *Section: 14.3*

Source: Loveless, Winter 2024 Exam 2, Problem 1(b). Similar practice: HW 14.3; Week 6 Tuesday WS.

Find $f_x(x,y)$ and $f_y(x,y)$ for $f(x,y) = y^x + y\sin(x^2y) + \ln x$.

$f_x(x,y) = y^x \ln(y) + 2xy^2\cos(x^2y) + \frac{1}{x}$

$f_y(x,y) = xy^{x-1} + \sin(x^2y) + x^2y\cos(x^2y)$

2. (4 pts) Implicit differentiation – *Sections: 14.3, 14.5*

Source: Loveless, Winter 2024 Exam 2, Problem 1(c). Similar practice: HW 14.3; Week 6 Tuesday WS and Thursday TP.

Given $z^3 + y^2z = x\sqrt{3+y^2}$, find z_y at $(x,y,z) = (1,1,1)$ using implicit differentiation.

$3z^2\frac{\partial z}{\partial y} + 2yz + y^2\frac{\partial z}{\partial y} = \frac{xy}{\sqrt{3+y^2}}$

at $(1,1,1)$ we have

$3\frac{\partial z}{\partial y} + 2 + \frac{\partial z}{\partial y} = \frac{1}{2}$

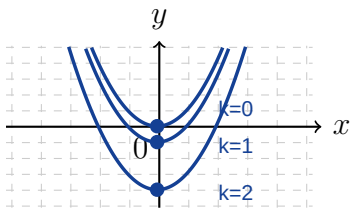
$\Rightarrow 4\frac{\partial z}{\partial y} = -\frac{3}{2}$

$\frac{\partial z}{\partial y} = -3/8$

3. (4 pts) Level curves – *Section: 14.1*

Modeled on: WebAssign HW 14.1, level-curve exercises. Similar practice: HW 14.1 contour/level-curve problems.

For $z = \sqrt{x^2 - y}$, sketch and label the level curves $z = k$ for $k = 0, 1, 2$ on the axes below. Label each curve and mark its y-intercept.



4. (12 pts) Gradients, directions, and the chain rule – *Sections: 14.5, 14.6*

Adapted from: Week 7 Tuesday WS #1 and Week 6 Thursday TP #2. Similar practice: HW 14.5 and HW 14.6.

Let

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2.$$

(a) [4 pts] Find $\nabla f(1,0)$ and the directional derivative of f at $(1,0)$ toward $(4,4)$.

(a) $f_x = 3y + 6x^2 + 9x$, $f_y = 3x - y$; hence $\nabla f(1,0) = \boxed{\langle 15, 3\rangle}$.

Toward $(4,4)$: $\mathbf{u} = \dfrac{\langle 3,4\rangle}{5}$, so

$$\boxed{D_{\mathbf{u}} f(1,0) = \nabla f(1,0)\cdot \mathbf{u} = \frac{45+12}{5} = \frac{57}{5}.}$$

(b) [3 pts] Find the maximum rate of increase of f at $(1,0)$ and a unit vector giving the direction in which it occurs.

(b) The maximum rate of increase is the length of the gradient:

$$\boxed{|\langle 15,3\rangle| = \sqrt{15^2+3^2} = 3\sqrt{26}.}$$

It occurs in the unit direction $\boxed{\mathbf{u} = \dfrac{\langle 15,3\rangle}{3\sqrt{26}} = \dfrac{\langle 5,1\rangle}{\sqrt{26}}.}$

(c) [5 pts] Suppose $z = f(x,y)$ with $x = s^2 + t$ and $y = st$. Use the chain rule to find z_s and z_t at $(s,t) = (1,2)$.

(c) At $(s,t) = (1,2)$, $x = s^2 + t = 3$ and $y = st = 2$.

$f_x(3,2) = 3(2) + 6(3)^2 + 9(3) = 87, \quad f_y(3,2) = 3(3) - 2 = 7.$

By the chain rule,

$$z_s = f_x x_s + f_y y_s = f_x(2s) + f_y t = 87(2) + 7(2) = \boxed{188},$$

$$z_t = f_x x_t + f_y y_t = f_x + f_y s = 87 + 7 = \boxed{94}.$$

5. (10 pts) Find and classify – *Section: 14.7*

Source: Loveless, Spring 2023 Exam 2, Problem 2(a). Similar practice: HW 14.7; Week 7 Tuesday Worksheet.

Find and classify all critical points of $f(x,y) = 6x^2 - 3x^2y + y^3 - 3y$. Show the second derivative test. You do not need the function values at the critical points.

$f_x = 12x - 6xy \stackrel{?}{=} 0 \rightarrow 6x(2-y) = 0 \Rightarrow x = 0$ or $y = 2$

$f_y = -3x^2 + 3y^2 - 3 \stackrel{?}{=} 0$

$x = 0 \Rightarrow 3y^2 - 3 = 0 \Rightarrow y = \pm 1$

$y = 2 \Rightarrow -7x^2 + 12 - 3 = 0 \Rightarrow 3x^2 = 9 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

$f_{xx} = 12 - 6y$

$f_{yy} = 6y$

$f_{xy} = -6x$

$(0,1) \Rightarrow D = (6)(6) - (0)^2 = 12 > 0$ MIN

$(0,-1) \Rightarrow D = (18)(-6) - (0)^2 < 0$ SADDLE

$(-\sqrt{3},2) \Rightarrow D = (0)(12) - (6\sqrt{3})^2 < 0$ SADDLE

$(\sqrt{3},2) \Rightarrow D = (0)(12) - (6\sqrt{3})^2 < 0$ SADDLE

List of Critical Points: MIN: $(0,1)$ SADDLE POINTS: $(0,-1)$, $(-\sqrt{3},2)$, $(\sqrt{3},2)$

6. (10 pts) Lagrange multipliers – *Section: 14.8*

Source: WebAssign HW 14.8, Problem 3 (Stewart 14.8.005). Similar practice: HW 14.8 Problems 2–3.

Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = xy$, subject to $16x^2 + y^2 = 32$.

Problem 6. Set $f(x,y) = xy$ and $g(x,y) = 16x^2 + y^2 = 32$.

$\nabla f = \lambda \nabla g$ gives $y = 32\lambda x$ and $x = 2\lambda y$. For $xy \neq 0$, $1 = 64\lambda^2$, so $\lambda = \pm\frac{1}{8}$ and $y = \pm 4x$.

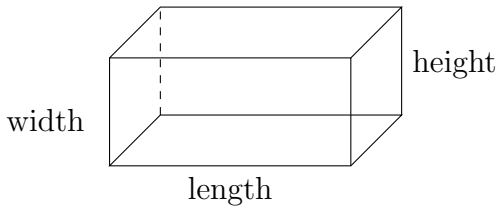
On $16x^2 + y^2 = 32$, the points are $(1,4), (-1,-4), (1,-4), (-1,4)$. Thus

Maximum 4 and minimum -4.

7. (14 pts) Rectangular mailing package – *Section: 14.8*

Source: WebAssign Optimization Applications HW, Problem 8 (Stewart 14.8.054).

A rectangular mailing box must have length plus girth (the perimeter of its cross-section perpendicular to the length) at most 114 inches. Find the dimensions of the box with the greatest volume. Explain why these dimensions give a maximum. (Length is the longest dimension.)



Approach 1: Substitution. Let L be the length and w, h the cross-section dimensions. At a maximum, the constraint is tight: $L + 2w + 2h = 114$, and $V = Lwh$. Thus

$$V(w, h) = (114 - 2w - 2h)wh.$$

Setting the first partial derivatives equal to zero gives

$$V_w = h(114 - 4w - 2h) = 0, \qquad V_h = w(114 - 2w - 4h) = 0.$$

For $w, h > 0$, this means $2w + h = 57$ and $w + 2h = 57$, so $w = h = 19$, and $L = 114 - 2(19) - 2(19) = 38$.

$$\boxed{L = 38 \text{ in}, \quad w = h = 19 \text{ in}, \quad V_{\max} = 13{,}718 \text{ in}^3.}$$

At the boundary of the feasible positive region $V = 0$; the positive interior critical point therefore gives the absolute maximum.

Approach 2: Lagrange multipliers. Maximize $V(L, w, h) = Lwh$ subject to $g(L, w, h) = L + 2w + 2h = 114$. Then $\nabla V = \lambda \nabla g$ means

$$wh = \lambda, \qquad Lh = 2\lambda, \qquad Lw = 2\lambda.$$

Since $L, w, h > 0$, these equations imply $L = 2w = 2h$. Using the constraint gives $3L = 114$, so

$$\boxed{L = 38, \quad w = h = 19, \quad V_{\max} = 38(19)^2 = 13{,}718.}$$

Again, boundary packages have volume 0, so this positive critical point is the absolute maximum.