1. (12 pts) Tangent plane and approximation – *Sections: 14.3, 14.4*

Source: Ostroff, Autumn 2023 Exam 2, Problem 4. Similar practice: HW 14.4; Week 6 Thursday TP.

Find the tangent plane to $z = xy^2 - \sqrt{x} - 3\sin(y-2)$ at $(4, 2, 14)$. Then use your plane to approximate y satisfying

$$14.22 = 3.92y^2 - \sqrt{3.92} - 3\sin(y-2)$$

.

$\frac{\partial z}{\partial x} = y^2 - \frac{1}{2\sqrt{x}}$ $\quad x=4,\ y=2 \rightarrow \quad \frac{\partial z}{\partial x} = \frac{15}{4}$

$\frac{\partial z}{\partial y} = 2xy - 3\cos(y-2)$ $\quad \frac{\partial z}{\partial y} = 13$

$$z = \frac{15}{4}(x-4) + 13(y-2) + 14$$

z x x

$14.22 = \frac{15}{4}(3.92-4) + 13(y-2) + 14$

$0.22 = -0.3 + 13(y-2)$

$0.52 = 13(y-2)$

$0.04 = y-2$

$y = 2.04$

2. (4 pts) Implicit partial derivative – *Sections: 14.3, 14.5*

Source: Week 6 Thursday Test Prep, Problem 4 (from Loveless Spring 2019). Similar practice: HW 14.5.

If $e^{2z} = x + y^2 z$ defines $z = z(x, y)$, find $\partial z/\partial y$ in terms of x, y, z.

Note: In Section 14.5, you will learn another way to do this with gradients, when you learn that come back and try this again!

$2e^{2z} \cdot \frac{\partial z}{\partial y} = 0 + 2yz + y^2 \frac{\partial z}{\partial y}$

$\Rightarrow (e^{2z} - y^2)\frac{\partial z}{\partial y} = 2yz$

$$\frac{\partial z}{\partial y} = \frac{2yz}{2e^{2z} - y^2}$$

OTHER METHOD

WRITE $F(x,y,z) = 0$, SO $e^{2z} - x - y^2 z = 0$

FIND $F_x = -1$, $F_y = -2yz$, $F_z = 2e^{2z} - y^2$

IT TURNS OUT THAT

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{2yz}{2e^{2z} - y^2}$$

3. (6 pts) Tangent line in a surface cross-section – *Sections: 14.3, 14.4*

Source: WebAssign HW 14.3 (Part 2), Problem 10 (Stewart 14.3.094).

The surface $z = 7 - x - x^2 - 3y^2$ meets the plane $x = 2$ in a parabola. Find parametric equations for the line tangent to this parabola at $(2, 3, -26)$.

Solution 1 (parametrize the cross-section). Because $x = 2$, the curve has $z = 7-2-2^2-3y^2 = 1-3y^2$. Taking $y = 3 + t$, the tangent vector at $y = 3$ is $\langle 0, 1, -6(3)\rangle = \langle 0, 1, -18\rangle$. Thus

$$\boxed{x = 2, \qquad y = 3 + t, \qquad z = -26 - 18t.}$$

Solution 2 (slope in the y-direction). Since $x = 2$ is fixed, y is the only remaining variable. Along the cross-section, $\dfrac{dz}{dy} = -6y$, which is -18 at $(2, 3, -26)$. So the slope in the y-direction is -18. Moving along the tangent line, $dx = 0$ and the tangent vector is $\langle 0, 1, -18\rangle$. Therefore the line is $\boxed{(x, y, z) = (2, 3, -26) + t\langle 0, 1, -18\rangle}$.

4. (5 pts) Temperature and gradient – *Section: 14.6*

Source: Loveless, Winter 2012 Math 324 Exam 2, Problem 5. Similar practice: HW 14.6; Week 7 Tuesday Worksheet.

For $T(x, y) = \frac{1}{3}x^2y + 5\sqrt{x^2 + y^2}$ in degrees Celsius (x, y in feet), find $\nabla T(3, 4)$ and the directional derivative in the direction $\langle -1, 2\rangle$. Give units.

Solution. $T_x = \frac{2}{3}xy + \frac{5x}{\sqrt{x^2+y^2}}$, $T_y = \frac{1}{3}x^2 + \frac{5y}{\sqrt{x^2+y^2}}$.

At $(3, 4)$, $\boxed{\nabla T = \langle 11, 7\rangle}$ (degrees C per foot).

For $\mathbf{u} = \langle -1, 2\rangle/\sqrt{5}$, $\boxed{D_{\mathbf{u}}T = \dfrac{-11 + 14}{\sqrt{5}} = \dfrac{3}{\sqrt{5}}}$ degrees C per foot.

5. (10 pts) Find and classify – *Section: 14.7*

Source: Ostroff, Autumn 2021 Exam 2, Problem 3. Similar practice: HW 14.7; Week 7 Tuesday Worksheet.

Find and classify all critical points of $f(x, y) = xy^2 - 2xy + x^2$ using the second derivative test.

$f_x(x,y) = y^2 - 2y + 2x = 0$

$f_y(x,y) = 2xy - 2x = 0 \rightarrow 2x(y-1) = 0$

$x = 0$ or $y = 1$

$y^2 - 2y = 0$

$y(y-2) = 0$

$y = 0$ or $y = 2$

$1 - 2 + 2x = 0$

$x = \frac{1}{2}$

Three critical points: $(0,0)$, $(0,2)$, and $(\frac{1}{2}, 1)$

$f_{xx}(x,y) = 2$

$f_{yy}(x,y) = 2x$

$f_{xy}(x,y) = 2y - 2$

$D(0,0) = 0 - (-2)^2 = -4 < 0$: saddle point @ $(0,0)$

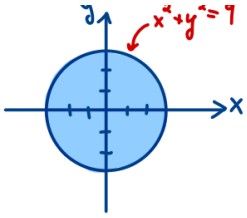
$D(0,2) = 0 - (2)^2 = -4 < 0$: saddle point @ $(0,2)$

$D(\frac{1}{2}, 1) = 2 \cdot 1 - 0^2 = 2 > 0$: local min @ $(\frac{1}{2}, 1)$

6. (11 pts) Global maximum and minimum on a disk – *Section: 14.7*

Source: Ostroff, Autumn 2023 Exam 2, Problem 3. Similar practice: HW 14.7; Week 7 Thursday Test Prep.

Find the absolute maximum and minimum values of $f(x, y) = x^2 + 4x + 2y^2$ on the closed disk $D = \{(x, y) : x^2 + y^2 \leq 9\}$. Check the interior and the boundary, and state where both extrema occur.



Crit pts: $f_x(x,y) = 2x + 4 = 0 \rightarrow x = -2$

$f_y(x,y) = 4y = 0 \rightarrow y = 0$

Only crit. pt. is $(-2,0)$

Boundary: $x^2 + y^2 = 9 \rightarrow y^2 = 9 - x^2$

So $f(x,y) = x^2 + 4x + 2(9 - x^2) = -x^2 + 4x + 18$

on domain $-3 \leq x \leq 3$.

Check endpoints $(-3,0)$ & $(3,0)$

$f'(x) = -2x + 4 = 0$

$x = 2$, so $2^2 + y^2 = 9$

$y = \pm\sqrt{5}$

Points to check

$(-2,0)$

$(-3,0)$

$(3,0)$

$(2,\sqrt{5})$

$(2,-\sqrt{5})$

$f(-2,0) = -4$ ← min

$f(-3,0) = -3$

$f(3,0) = 21$

$f(2,\pm\sqrt{5}) = 22$ ← max

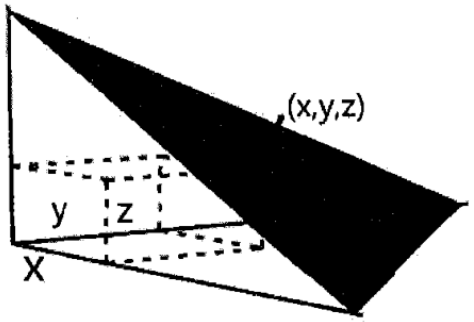
Problem 6: See the last page for a different approach using Lagrange multipliers.

7. (12 pts) Maximum-volume box – *Sections: 14.7, 14.8*

Source: Historical Math 126 Exam 2, first-octant box question (archived with old solutions). Similar practice: Optimization Applications HW.

Find the dimensions of the rectangular box of maximum volume in the first octant, with three faces in the coordinate planes and its opposite vertex (x, y, z) on $4x + 3y + z = 12$. Use the second derivative test to verify the critical point gives a local maximum.

MAXIMIZE
VOLUME OF BOX $= xyz$
SUBJECT TO CONSTRAINT
$z = 12 - 4x - 3y$



$\text{VOL} = f(x,y) = 12xy - 4x^2y - 3xy^2$

$f_x = 12y - 8xy - 3y^2 \stackrel{?}{=} 0 \Rightarrow y(12 - 8x - 3y) \stackrel{?}{=} 0$

$f_y = 12x - 4x^2 - 6xy \stackrel{?}{=} 0 \quad 2x(6 - 2x - 3y) \stackrel{?}{=} 0$

$12 - 8x - 3y \stackrel{?}{=} 0 \Rightarrow y = \frac{12 - 8x}{3}$

COMBINE $6 - 2x - 3\left(\frac{12 - 8x}{3}\right) \stackrel{?}{=} 0$

$6 - 2x - 12 + 8x \stackrel{?}{=} 0 \Rightarrow 6x = 6 \Rightarrow \boxed{x = 1}$

$\boxed{y = 4/3}$

$\boxed{z = 12 - 4 - 4}$

$\boxed{(1, \tfrac{4}{3}, 4) = (x, y, z)}$

2ND DERIV. TEST

$f_{xx} = -8y$
$f_{yy} = -6x$
$f_{xy} = 12 - 8x - 6y$

$f_{xx}(1, 4/3) = -\frac{32}{3} < 0$
$f_{yy}(1, 4/3) = -6 < 0$
$f_{xy}(1, 4/3) = 12 - 8 - 8 = -4$

LOCAL MAX!!!

$D = \left(-\frac{32}{3}\right)(-6) - (-4)^2 = 64 - 16 = 48 > 0$

Problem 7: See the last page for a different approach using Lagrange multipliers.

Sample B — Alternative Solutions Using Lagrange Multipliers

Problems 6 and 7 (additional methods; the original handwritten solutions remain on the preceding pages)

Problem 6. Absolute extrema on the disk. $f(x,y) = x^2 + 4x + 2y^2$, $x^2 + y^2 \le 9$.

Step 1: Check the interior. Here $\nabla f = \langle 2x+4, 4y\rangle = \mathbf{0}$ gives $(x,y) = (-2,0)$, which lies in the disk, with $f(-2,0) = -4$.

Step 2: Use Lagrange multipliers on the boundary $g(x,y) = x^2 + y^2 = 9$.

$$\nabla f = \lambda \nabla g \quad \Longleftrightarrow \quad 2x + 4 = 2\lambda x, \qquad 4y = 2\lambda y, \qquad x^2 + y^2 = 9.$$

If $y = 0$, then $x = \pm 3$. This gives $f(3,0) = 21$ and $f(-3,0) = -3$.

If $y \neq 0$, then $\lambda = 2$. The first equation becomes $2x+4 = 4x$, so $x = 2$. Thus $y = \pm\sqrt{5}$, with $f(2, \pm\sqrt{5}) = 22$.

The disk is closed and bounded, so the absolute extrema occur among these candidates:

$$\boxed{\text{Absolute minimum } -4 \text{ at } (-2,0).} \qquad \boxed{\text{Absolute maximum } 22 \text{ at } (2, \pm\sqrt{5}).}$$

Problem 7. Maximum-volume box. Maximize $V(x,y,z) = xyz$ with $4x + 3y + z = 12$ and $x, y, z \ge 0$.

Step 1: Apply Lagrange multipliers. With $g(x,y,z) = 4x + 3y + z$,

$$\nabla V = \lambda \nabla g \quad \Longleftrightarrow \quad yz = 4\lambda, \qquad xz = 3\lambda, \qquad xy = \lambda, \qquad 4x + 3y + z = 12.$$

For an interior maximum $x, y, z > 0$, dividing by $xy = \lambda$ gives $z = 4x$ and $z = 3y$, so $y = \frac{4}{3}x$. The constraint becomes $4x + 4x + 4x = 12$, hence

$$\boxed{x = 1, \qquad y = \tfrac{4}{3}, \qquad z = 4, \qquad V_{\max} = \tfrac{16}{3}.}$$

Step 2: Verify the second-derivative test requested in the question. Substitute $z = 12 - 4x - 3y$ to obtain $h(x,y) = xy(12 - 4x - 3y)$. At $(1, \frac{4}{3})$,

$$h_{xx} = -8y = -\tfrac{32}{3} < 0, \quad h_{yy} = -6x = -6, \quad h_{xy} = 12 - 8x - 6y = -4,$$

$$h_{xx}h_{yy} - h_{xy}^2 = \left(-\tfrac{32}{3}\right)(-6) - (-4)^2 = 48 > 0.$$

Thus h has a strict local maximum. Any boundary box has zero volume, while this is the only positive interior candidate, so it is also the **absolute maximum**.