WORKED SOLUTIONS

1. (12 pts) – *Sections: 14.3, 14.4*

Source: Ostroff, Spring 2019 Math 126 Exam 2, Problem 1(a)–(b). Similar practice: HW 14.4; Week 6 TP #1.

Consider the function $f(x,y) = xy^2 + \sqrt{y-x} - e^{x-2}$.

(a) **[8 pts]** Find the linearization $L(x,y)$ of f at $(2,3)$.

$f_x(x,y) = y^2 - \frac{1}{2\sqrt{y-x}} - e^{x-2}$

$f_x(2,3) = 9 - \frac{1}{2} - 1 = 7.5$

$f_y(x,y) = 2xy + \frac{1}{2\sqrt{y-x}}$

$f_y(2,3) = 12 + \frac{1}{2} = 12.5$

$f(2,3) = 18 + 1 - 1 = 18$

$L(x,y) = 7.5(x-2) + 12.5(y-3) + 18$

or

$L(x,y) = 7.5x + 12.5y - 34.5$

(b) **[4 pts]** Use your answer from part (a) to find an approximate solution for x to

$$9.61x + \sqrt{3.1 - x} - e^{x-2} = 19.1.$$

$L(x, 3.1) = 19.1$

$7.5x + 12.5(3.1) - 34.5 = 19.1$

$x = 1.98$

WORKED SOLUTIONS

2. (6 pts) Implicit differentiation – *Sections: 14.3, 14.5*

Source: Week 6 Tuesday Worksheet, Problem 2 (shortened). Similar practice: HW 14.5; Week 6 Thursday TP #4.

The surface S is defined implicitly by $\sqrt{x^2+z} - xy + ze^{y-3} = 2$. Near $P = (2,3,5)$, suppose $z = g(x,y)$. Find $\dfrac{\partial z}{\partial x}$ at P. Show your work.

$\frac{\partial z}{\partial x}$: REWRITE: $(x^2+z)^{1/2} - xy + ze^{y-3} = 2$

$\frac{1}{2}(x^2+z)^{-1/2}\cdot(2x+z_x) - y + z_x e^{y-3} = 0$

AT $(x,y,z) = (2,3,5)$

WE HAVE $\frac{1}{2}(4+5)^{-1/2}(4+z_x) - 3 + z_x e^0 = 0$

$\cdot 6 \;\Rightarrow\; \frac{1}{6}(4+z_x) - 3 + z_x = 0$

$4 + z_x - 18 + 6z_x = 0 \quad 7z_x = 14 \quad z_x = 2$

3. (6 pts) Chain rule along a curve – *Section: 14.5*

Source: Loveless, Winter 2012 Math 324 Exam 2, Problem 2(b). Similar practice: HW 14.5 #3–4, #8.

A particle moves along a curve on the surface $z = x^2 + \sin y - xy^2$. Suppose $x(1) = 2$, $y(1) = 0$, $x'(1) = 3$, and $y'(1) = -5$. Find $\dfrac{dz}{dt}$ at $t = 1$.

$$z_x = 2x - y^2, \qquad z_y = \cos y - 2xy$$
$$\frac{dz}{dt} = z_x\frac{dx}{dt} + z_y\frac{dy}{dt}$$
$$\left.\frac{dz}{dt}\right|_{t=1} = 4(3) + 1(-5) = \mathbf{7}$$

4. (3 pts) Gradient and steepest increase – *Section: 14.6*

Adapted from: WebAssign HW 14.6, #7 (Stewart 14.6.029.MI). Similar practice: Week 7 Tuesday WS #1.

For $f(x,y) = 7\sin(x+2y)$, find the maximum rate of increase at $\left(\frac{\pi}{6}, \frac{\pi}{12}\right)$ and a unit vector giving the direction in which it occurs.

$$\nabla f(x,y) = 7\cos(x+2y)\langle 1,2\rangle$$
$$x + 2y = \pi/3, \quad \nabla f = \tfrac{7}{2}\langle 1,2\rangle$$
$$\text{Max rate} = \tfrac{7\sqrt{5}}{2}, \quad \mathbf{u} = \tfrac{\langle 1,2\rangle}{\sqrt{5}}$$

WORKED SOLUTIONS

5. (11 pts) – *Section: 14.7*

Source: Ostroff, Autumn 2023 Math 126 Exam 2, Problem 1. Similar practice: HW 14.7; Week 7 Tuesday WS #2.

Find all critical points of $f(x,y) = 6xy + x^3 - y^2$ and classify each as a local maximum, local minimum, or saddle point. Use the second derivative test.

$f_x(x,y) = 6y + 3x^2 = 0 \rightarrow 6(3x) + 3x^2 = 0 \rightarrow 3x(6+x) = 0$

$f_y(x,y) = 6x - 2y = 0 \rightarrow y = 3x$

$x = 0$ or $x = -6$

$y = 0$ $\quad$ $y = -18$

$(0,0)$ & $(-6,-18)$

$f_{xx}(x,y) = 6x$

$f_{yy}(x,y) = -2$

$f_{xy}(x,y) = 6$

$D(0,0) = 0 - 6^2 < 0$, so saddle point @ $(0,0)$

$D(-6,-18) = (-36)(-2) - 6^2 > 0$ and $-36 < 0$, so local max @ $(-6,-18)$

6. (10 pts) – *Section: 14.8*

Source: WebAssign HW 14.8, #2 (Stewart 14.8.003). Similar practice: HW 14.8 #2–3.

Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x^2 - y^2$ subject to $x^2 + y^2 = 4$.

$\nabla f = \langle 2x, -2y \rangle$, $\nabla g = \langle 2x, 2y \rangle$ for $g = x^2 + y^2 = 4$.

$2x = 2\lambda x$, $-2y = 2\lambda y$, $x^2 + y^2 = 4$.

If $x \neq 0$: $\lambda = 1$, $y = 0$, giving $(\pm 2, 0)$ and $f = 4$.

If $y \neq 0$: $\lambda = -1$, $x = 0$, giving $(0, \pm 2)$ and $f = -4$.

Absolute maximum value: $\boxed{4}$ at $(\pm 2, 0)$.

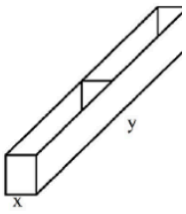
Absolute minimum value: $\boxed{-4}$ at $(0, \pm 2)$.

7. (12 pts) – *Sections: 14.7, 14.8*

Source: Loveless, Spring 2023 Math 126 Exam 2, Problem 4. Similar practice: HW 14.7–14.8 Optimization Applications #5–6.

Your job is to design a box with a bottom, four side walls, one dividing wall in the middle, and no top. All walls are made of plywood. The total volume must be 6 cubic feet, and plywood costs 3 per square foot.

Find the minimum cost to produce the box. You *must* give a two-variable cost function and find its critical point. You do not have to perform the second derivative test.



COST = 5 · AREA

$= 3 \cdot (xy + 2yz + 3xz)$

CONSTRAINT: $xyz = 6 \Rightarrow z = \frac{6}{xy}$

$\text{COST} = f(x,y) = 3xy + 6y\frac{6}{xy} + 9x\frac{6}{xy}$

$f(x,y) = 3xy + \frac{36}{x} + \frac{54}{y}$

$f_x = 3y - \frac{36}{x^2} \overset{?}{=} 0 \Rightarrow y = \frac{12}{x^2}$

$f_y = 3x - \frac{54}{y^2} \overset{?}{=} 0 \rightarrow 3xy^2 - 54 = 0$

$\rightarrow xy^2 - 18 = 0$

$x\left(\frac{12}{x^2}\right)^2 - 18 = 0$

$\frac{144}{x^3} - 18 = 0$

$x^3 = \frac{144}{18} = 8$

$\boxed{x = 2} \Rightarrow \boxed{y = \frac{12}{(2)^2} = \frac{12}{4} = 3} \Rightarrow z = \frac{6}{xy} = 1$

MIN COST $= 3 \cdot (6 + 6 + 6) = 3 \cdot 18 = \boxed{\$54}$

You may use this page for scratch-work or extra room.

All work on this page will be ignored unless you clearly indicate on the relevant problem that additional work appears here, and label that work with the problem number.