Partial Derivatives Examples And A Quick Review of Implicit Differentiation

Given a multi-variable function, we defined the partial derivative of one variable with respect to another variable in class. All other variables are treated as constants.

Here are some basic examples:

1. If $z = f(x, y) = x^4 y^3 + 8x^2 y + y^4 + 5x$, then the partial derivatives are

$$\frac{\partial z}{\partial x} = 4x^3 y^3 + 16xy + 5 \qquad \text{(Note: } y \text{ fixed, } x \text{ independent variable, } z \text{ dependent variable)}$$

$$\frac{\partial z}{\partial y} = 3x^4 y^2 + 8x^2 + 4y^3 \qquad \text{(Note: } x \text{ fixed, } y \text{ independent variable, } z \text{ dependent variable)}$$

2. If $z = f(x, y) = (x^2 + y^3)^{10} + \ln(x)$, then the partial derivatives are

$$\frac{\partial z}{\partial x} = 20x(x^2 + y^3)^9 + \frac{1}{x} \qquad \text{(Note: We used the chain rule on the first term)}$$

$$\frac{\partial z}{\partial y} = 30y^2(x^2 + y^3)^9 \qquad \text{(Note: Chain rule again, and second term has no } y\text{)}$$

3. If $z = f(x, y) = xe^{xy}$, then the partial derivatives are

$$\frac{\partial z}{\partial x} = e^{xy} + xye^{xy} \qquad \text{(Note: Product rule (and chain rule in the second term)}$$

$$\frac{\partial z}{\partial y} = x^2 e^{xy} \qquad \text{(Note: No product rule, but we did need the chain rule)}$$

4. If $w = f(x, y, z) = \frac{y}{x+y+z}$, then the partial derivatives are

$$\frac{\partial w}{\partial x} = \frac{(x+y+z)(0) - (1)(y)}{(x+y+z)^2} = \frac{-y}{(x+y+z)^2} \qquad \text{(Note: Quotient Rule)}$$

$$\frac{\partial w}{\partial y} = \frac{(x+y+z)(1) - (1)(y)}{(x+y+z)^2} = \frac{x+z}{(x+y+z)^2} \qquad \text{(Note: Quotient Rule)}$$

$$\frac{\partial w}{\partial z} = \frac{(x+y+z)(0) - (1)(y)}{(x+y+z)^2} = \frac{-y}{(x+y+z)^2} \qquad \text{(Note: Quotient Rule)}$$

Aside: We actually only needed the quotient rule for $\frac{\partial w}{\partial y}$, but I used it in all three to illustrate that the differences (and to show that it can be used even if some derivatives are zero).

If you are forgetting your derivative rules, here are the most basic ones again (the general exponential rule $\frac{d}{dx}(a^x) = a^x \ln(a)$ appears in one homework problem):

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(e^x) = e^x, \ \frac{d}{dx}(a^x) = a^x \ln(a)$	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$
$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$	$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$	$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$	$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{x^2+1}$
$(FS)' = FS' + F'S$	$\left(\frac{N}{D}\right)' = \frac{DN' - ND'}{D^2}$	$[f(g(x))]' = f'(g(x))g'(x)$

There are some situations when we have an equation implicitly defining a surface (meaning it is not of the form $z = f(x, y)$, with z by itself on one side). In Math 124, you discussed how to find derivatives in this situation using what is called *implicit differentiation*. The basic observation is this:

If z is an implicit function of x (that is, z is a dependent variable in terms of the independent variable x), then we can use the chain rule to say what derivatives of z should look like. For example, if $z = \sin(x)$, and we want to know what the derivative of z^2, then we can use the chain rule. $\frac{d}{x}(z^2) = 2z\frac{dz}{dx} = 2\sin(x)\cos(x)$. In real situations where we use this, we don't know the function z, but we can still write out the second step in this process from above and then solve for $\frac{dz}{dx}$. So for example, if y is a function of x, then the derivative of $y^4 + x + 3$ with respect to x would be $4y^3\frac{dy}{dx} + 1$.

Here are some Math 124 problems pertaining to implicit differentiation (these are problems directly from a practice sheet I give out when I teach Math 124).

1. Given $x^4 + y^4 = 3$, find $\frac{dy}{dx}$.
   ANSWER: Differentiating with respect to x (and treating y as a function of x) gives

   $$4x^3 + 4y^3\frac{dy}{dx} = 0 \quad \text{(Note the chain rule in the derivative of } y^4\text{)}$$

   Now we solve for $\frac{dy}{dx}$, which gives

   $$\frac{dy}{dx} = \frac{-x^3}{y^3}.$$

   Note that we get both x's and y's in the answer, but at least we get some answer.

2. Given $y^3 - x^2y - 2x^3 = 8$, find $\frac{dy}{dx}$
   ANSWER: Differentiating with respect to x (and treating y as a function of x) gives

   $$3y^2\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} - 6x^2 = 0 \quad \text{(We used the product rule in the middle term)}$$

   Now we solve for $\frac{dy}{dx}$, which gives

   $$(3y^2 - x^2)\frac{dy}{dx} = 6x^2 + 2xy, \text{ so } \frac{dy}{dx} = \frac{6x^2 + 2xy}{3y^2 - x^2}.$$

   The solving step can sometimes take a bit of algebra in the end to clean up your answer.

Students who remember implicit differentiation sometimes ask why we aren't implicitly differentiating y when we are taking the derivative with respect to x in a multivariable function. And the answer is: It depends on the role the variable is playing.

When we are taking a partial derivative all variables are treated as fixed constant except two, the independent variable and the dependent variable.

Let's do some examples:

1. Given $x^2 + \cos(y) + z^3 = 1$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
   ANSWER: Differentiating with respect to x (and treating z as a function of x, and y as a constant) gives

   $$2x + 0 + 3z^2\frac{\partial z}{\partial x} = 0 \quad \text{(Note the chain rule in the derivative of } z^3\text{)}$$

   Now we solve for $\frac{\partial z}{\partial x}$, which gives

   $$\frac{\partial z}{\partial x} = \frac{-2x}{3z^2}.$$

Note that we get z's in the answer, but, as before, at least we get some answer.

Now for $\frac{\partial z}{\partial y}$. Differentiating with respect to y (and treating z as a function of y, and x as a constant) gives

$$0 - \sin(y) + 3z^2 \frac{\partial z}{\partial y} = 0 \quad \text{and solving gives} \quad \frac{\partial z}{\partial y} = \frac{\sin(y)}{3z^2}.$$

2. Given $\sin(xyz) = x + 3z + y$, find $\frac{\partial z}{\partial x}$

   ANSWER: Differentiating with respect to x (and treating z as a function of x, and y as a constant) gives

$$\cos(xyz)(yz + xy\frac{\partial z}{\partial x}) = 1 + 3\frac{\partial z}{\partial x} \quad \text{(Note the use of the product and chain rules)}$$

   Now we expand and solve for $\frac{\partial z}{\partial x}$, which gives

$$yz \cos(xyz) + xy \cos(xyz)\frac{\partial z}{\partial x} = 1 + 3\frac{\partial z}{\partial x}$$

$$(xy \cos(xyz) - 3)\frac{\partial z}{\partial x} = 1 - yz \cos(xyz)$$

$$\frac{\partial z}{\partial x} = \frac{1 - yz \cos(xyz)}{xy \cos(xyz) - 3}.$$

I hope this sheet reminds you of some of the finer points of differentiating and helps to clarify partial derivatives.