Applied Max/Min

A collection of optimization problems for Math 126 (Calculus III)

Optimization problems are everywhere! Some are easy to solve, some are surprisingly difficult, and some remain unsolved. The goal is to explore interesting questions anyone can understand, even when the mathematics needed to solve them is far from simple.

For each problem:

- What do you want (objective)? Write a function such as $z = f(x, y)$.
- What is given (constraint)? Write an equation such as $g(x, y) = c$.

Dr. Loveless's Optimization Videos: General Tips · Precalculus · Calculus I Examples

HW (Fall 2026) (Math 126 – Calculus III)

1. Find the point on the plane $x - 2y + 3z = 9$ closest to $(0, 2, 1)$.
2. Find the points on the cone $z^2 = x^2 + y^2$ closest to $(6, 2, 0)$.
3. Find the points on $y^2 = 49 + xz$ closest to the origin.

   Explore: Dr. Loveless Curiosity Lab: Minimum Distance
4. Find the largest volume of a rectangular box in the first octant, with three faces on the coordinate planes and a vertex on $x + 2y + 3z = 3$.

   Explore: Dr. Loveless Video: Box Under a Plane
5. An open-top aquarium has volume V, a slate base, and glass sides. Slate costs seven times as much per unit area as glass. Find the dimensions that minimize material cost.
6. An open-top cardboard box has volume 27,436 cm^3. Find its dimensions that minimize the amount of cardboard.

   Explore: Dr. Loveless Video: Building a Bento Box
7. A cylindrical mailing tube must have length plus girth at most 96 inches. (Girth is the circumference perpendicular to its length.) Find the radius and length that maximize volume.
8. A rectangular mailing box must have length plus girth at most 114 inches. (Girth is the perimeter perpendicular to its length.) Find the length, width, and height that maximize volume.

Other Classic Examples — Geometry and Applications

9. **Rectangle in an Ellipse.** (Math 124 – Calculus I) Find the largest area of a rectangle with sides parallel to the axes inscribed in $\frac{x^2}{9} + \frac{y^2}{4} = 1$.
10. **Box in an Ellipsoid.** (Math 126 – Extension) Find the largest volume of a rectangular box with faces parallel to the coordinate planes inscribed in $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1$.
11. **Rectangle in a Triangle.** (Math 124 – Calculus I) A right triangle has legs of lengths 4 cm and 5 cm. Find the largest rectangle with two sides along the legs of the triangle.
12. **Cone in a Sphere.** Find the dimensions of the largest right circular cone that fits inside a sphere of radius R.
13. **Cylinder in a Sphere.** Find the dimensions of the largest right circular cylinder that fits inside a sphere of radius R.
14. **Cylinder in a Cone.** A cone has height 12 inches and base radius 6 inches. Find the largest-volume cylinder that fits inside the cone, with the same axis and its base on the cone's base.
15. **Splitting a Wire.** (Math 124 – Calculus I) An 18-meter wire is cut into two pieces. One piece is bent into a square and the other into an equilateral triangle. How should the wire be divided to maximize the total area? How about to minimize it?

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16. **Cylindrical Tank.** A closed cylindrical tank must hold 1000 m^3 of water. Find the radius and height that minimize the amount of material used.

17. **Locating a Distribution Center.** Three factories are located at $(0, 0)$, $(6, 0)$, and $(0, 8)$. Find the point that minimizes the sum of the distances to the factories.

    (a) What if a fourth factory is added at $(12, 10)$?

    (b) What if there are n factories at $(a_1, b_1), \ldots, (a_n, b_n)$?

18. **Two Moving Boats.** (Math 124 – Calculus I) A boat leaves a dock at 6:00 PM and travels south at 20 km/h. Another boat travels east at 15 km/h and reaches the dock at 7:00 PM. When are the boats closest together?

19. **Line Cutting Off Area.** (Math 124 – Calculus I) Find the line through $(3, 5)$ that intersects the positive coordinate axes and cuts off the smallest area in the first quadrant.

20. **Plane Cutting Off Volume.** (Math 126 – Extension) A plane passes through (a, b, c), where $a, b, c > 0$, and intersects the three positive coordinate axes. Find the plane that cuts off the smallest volume in the first octant.

21. **Box Under a Paraboloid.** (Math 126 – Extension) Find the dimensions of the largest rectangular box in the first octant, with three faces on the coordinate planes and its opposite vertex on $z = 12 - x^2 - y^2$.

22. **Rowing and Running.** (Math 124 – Calculus I) A river is 3 km wide. A person wants to reach a point 3 km downstream on the opposite bank. They can row at 6 km/h and run at 8 km/h. Where should they land to arrive as quickly as possible?

    Explore: Dr. Loveless Curiosity Lab: Rowing and Running

23. **Walking or Rowing Around a Lake.** (Math 124 – Calculus I) A person is on the shore of a circular lake of radius 2 miles and wants to reach the diametrically opposite point. They can walk at 4 mph or row at 2 mph. What combination of walking and rowing minimizes travel time?

24. **Pipeline Across a River.** (Math 124 – Calculus I) An oil refinery is on the north bank of a river 2 km wide. Storage tanks are on the south bank, 6 km east of the refinery. Pipe costs \$300,000/km on land and \$500,000/km underwater. Find the least expensive route.

25. **Light-Rail Tunnel.** (Dr. Loveless Original) Engineers are designing a new light-rail route from a station at $(0, 0, 0)$ to a point $(3000, 4000, -100)$ beneath the Lake Washington Ship Canal, where the train must travel underground. Coordinates are in feet. Above-ground construction costs \$20,000 per foot and underground construction costs \$60,000 per foot. Assuming two straight segments meeting at $z = 0$, where should the train enter the tunnel to minimize cost?

26. **Light-Rail Grade Restriction.** (Dr. Loveless Original) Revisit the previous light-rail problem, but require the underground track to have a maximum grade of 5%. How does this change the optimal tunnel entrance?

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Going Further — Simple Questions, Surprising Mathematics

Some of these problems are extremely difficult, and some generalizations remain unsolved.

For each problem, try to identify what you want to maximize or minimize and what restrictions you have. This is just for fun and interest, some of these might not have a nice answer or may need more assumptions. Some may be unsolved or lead to deep math. This is just giving you more examples of optimization for exploration.

27. **Getting Around a Corner.**

    (a) (Math 124 – Calculus I) Two hallways of widths 9 ft and 6 ft meet at a right angle. What is the longest straight, rigid, thin pipe that can be carried horizontally around the corner?

    (b) What if both hallways have 8-ft ceilings and the pipe is allowed to tilt and rotate in three dimensions?

    (c) What is the largest-area rigid, flat shape that can be moved around a right-angle corner in a hallway of width 1? The shape may rotate but cannot bend. (The Moving Sofa Problem.)

    More: Moving Sofa — Recent Proof · Illustrated Article

28. **Spreading Out People.** (Dr. Loveless Classroom Exploration) When arranging students for an exam, or family members for Thanksgiving, how should we place them inside a unit square so that the minimum distance between any two people is as large as possible?

    (a) How would you arrange 3 people? What about 4, 5, or 6 people?

    (b) What if there are n people?

    (c) What if the region is a circle instead of a square?

    Explore: Dr. Loveless Video: Spreading Out Your Family at Thanksgiving

    Explore: Circle Packing and Related Problems

29. **Shortest Delivery Route.** A delivery driver must visit 10 homes and return to the starting point. In what order should the homes be visited to minimize total travel distance? What if there are 100 homes?

    Explore: The Traveling Salesperson Problem

30. **Where Should Fire Stations Go?** A city can build three new fire stations. Where should they be located to minimize the distance to the farthest home? What if the goal is to minimize the average distance instead?

    Explore: Facility Location Problems

31. **Fastest Roller Coaster.** A roller-coaster car starts from rest and moves only under gravity. What shape of track gets it from one point to another in the shortest time? Assume no friction.

    Explore: The Brachistochrone Problem

    Explore: Dr. Loveless Curiosity Lab: Roller Coaster Geometry

32. **Soap Bubbles and Minimum Surface Area.** (Dr. Loveless Project) Two circular wire rings, each of radius 1, are held parallel and one unit apart. What shape of soap film connecting the rings has the smallest surface area? What happens as the rings are pulled farther apart?

    Explore: Dr. Loveless: Surface Area and Soap Bubbles

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Objectives and Constraints

All dimensions are positive unless otherwise indicated. Distance objectives below use distance squared.

1. **Point on a plane**

$$\text{Minimize } D^2 = f(x,y,z) = x^2 + (y-2)^2 + (z-1)^2, \qquad g(x,y,z) = x - 2y + 3z = 9.$$

2. **Point on a cone**

$$\text{Minimize } D^2 = f(x,y,z) = (x-6)^2 + (y-2)^2 + z^2, \qquad g(x,y,z) = x^2 + y^2 - z^2 = 0.$$

3. **Point closest to the origin**

$$\text{Minimize } D^2 = f(x,y,z) = x^2 + y^2 + z^2, \qquad g(x,y,z) = y^2 - xz = 49.$$

4. **Box under a plane**

$$\text{Maximize } V = f(x,y,z) = xyz, \qquad g(x,y,z) = x + 2y + 3z = 3.$$

5. **Aquarium**

$$\text{Minimize } C = f(x,y,z) = 7xy + 2xz + 2yz, \qquad g(x,y,z) = xyz = V.$$

   Here x, y are the base dimensions, z is the height, and cost is measured in units of the glass cost per unit area.

6. **Open-top cardboard box**

$$\text{Minimize } A = f(x,y,z) = xy + 2xz + 2yz, \qquad g(x,y,z) = xyz = 27{,}436.$$

7. **Cylindrical mailing tube**

$$\text{Maximize } V = f(r,h) = \pi r^2 h, \qquad g(r,h) = h + 2\pi r = 96.$$

8. **Rectangular mailing box**

$$\text{Maximize } V = f(x,y,z) = xyz, \qquad g(x,y,z) = x + 2y + 2z = 114.$$

   Here x is the length.

9. **Rectangle in an ellipse**

$$\text{Maximize } A = f(x,y) = 4xy, \qquad g(x,y) = \frac{x^2}{9} + \frac{y^2}{4} = 1.$$

   Here x, y are half the side lengths.

10. **Box in an ellipsoid**

$$\text{Maximize } V = f(x,y,z) = 8xyz, \qquad g(x,y,z) = \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{25} = 1.$$

    Here x, y, z are half the side lengths.

11. **Rectangle in a triangle**

$$\text{Maximize } A = f(x,y) = xy, \qquad g(x,y) = \frac{x}{4} + \frac{y}{5} = 1.$$

12. **Cone in a sphere**

$$\text{Maximize } V = f(r,h) = \frac{1}{3}\pi r^2 h, \qquad g(r,h) = r^2 + (h-R)^2 = R^2.$$

    Here $0 < h < 2R$.

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13. **Cylinder in a sphere**

$$\text{Maximize } V = f(r,h) = \pi r^2 h, \qquad g(r,h) = r^2 + \frac{h^2}{4} = R^2.$$

14. **Cylinder in a cone**

$$\text{Maximize } V = f(r,h) = \pi r^2 h, \qquad g(r,h) = 2r + h = 12.$$

15. **Splitting a wire**

$$\text{Maximize or minimize } A = f(x,y) = x^2 + \frac{\sqrt{3}}{4}y^2, \qquad g(x,y) = 4x + 3y = 18.$$

Here x is the square's side length and y is the triangle's side length. Allow $x, y \geq 0$ to include endpoints.

16. **Closed cylindrical tank**

$$\text{Minimize } A = f(r,h) = 2\pi r^2 + 2\pi r h, \qquad g(r,h) = \pi r^2 h = 1000.$$

17. **Distribution center**

$$\text{Minimize } D = f(x,y) = \sqrt{x^2+y^2} + \sqrt{(x-6)^2+y^2} + \sqrt{x^2+(y-8)^2}.$$

No constraint equation; $(x,y) \in \mathbb{R}^2$.

(a) With the fourth factory:

$$D = f(x,y) = \sqrt{x^2+y^2} + \sqrt{(x-6)^2+y^2} + \sqrt{x^2+(y-8)^2} + \sqrt{(x-12)^2+(y-10)^2}.$$

(b) With n factories:

$$D = f(x,y) = \sum_{i=1}^{n} \sqrt{(x-a_i)^2+(y-b_i)^2}.$$

18. **Two moving boats**

$$\text{Minimize } D^2 = f(t) = 400t^2 + 225(t-1)^2, \qquad t \geq 0.$$

Here t is hours after 6:00 PM; no constraint equation.

19. **Line cutting off area**

$$\text{Minimize } A = f(x,y) = \frac{1}{2}xy, \qquad g(x,y) = \frac{3}{x} + \frac{5}{y} = 1.$$

Here x, y are the positive axis intercepts.

20. **Plane cutting off volume**

$$\text{Minimize } V = f(x,y,z) = \frac{1}{6}xyz, \qquad g(x,y,z) = \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1.$$

Here x, y, z are the positive axis intercepts.

21. **Box under a paraboloid**

$$\text{Maximize } V = f(x,y,z) = xyz, \qquad g(x,y,z) = x^2 + y^2 + z = 12.$$

22. **Rowing and running**

$$\text{Minimize } T = f(x,y) = \frac{x}{6} + \frac{y}{8}, \qquad g(x,y) = x^2 - (3-y)^2 = 9.$$

Here x is the rowing distance and y is the running distance, with $0 \leq y \leq 3$.

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23. **Walking or rowing around a lake**

$$\text{Minimize } T = f(x,\theta) = \frac{x}{2} + \frac{2(\pi-\theta)}{4}, \qquad g(x,\theta) = x - 4\sin(\theta/2) = 0.$$

Here x is the rowing distance and θ is its central angle, with $0 \le \theta \le \pi$.

24. **Pipeline across a river**

$$\text{Minimize } C = f(x,y) = 500{,}000x + 300{,}000y, \qquad g(x,y) = x^2 - (6-y)^2 = 4.$$

Here x is the underwater pipe length and y is the land pipe length, with $0 \le y \le 6$.

25. **Light-rail tunnel**

$$\text{Minimize } C = f(x,y,z) = 20{,}000\sqrt{x^2+y^2+z^2} + 60{,}000\sqrt{(3000-x)^2+(4000-y)^2+(-100-z)^2},$$
$$g(x,y,z) = z = 0.$$

Here (x,y,z) is the tunnel entrance.

26. **Light-rail grade restriction**
Minimize the same function $C = f(x,y,z)$ as in Problem 25, subject to

$$g(x,y,z) = z = 0, \qquad q(x,y) = (3000-x)^2 + (4000-y)^2 \ge 2000^2.$$

27. **Getting around a corner**

(a) The limiting pipe length at angle θ is

$$L = f(\theta) = 9\csc\theta + 6\sec\theta, \qquad 0 < \theta < \frac{\pi}{2}.$$

Minimize this function to find the longest pipe that can pass through every angle.

(b) Objective: maximize L.
Constraint: a continuous rigid motion of the pipe must stay inside the hallways and below the 8-ft ceilings.

(c) Objective: maximize $A = f(S) = \text{Area}(S)$.
Constraint: the rigid shape S must admit a continuous planar motion around the width-1 hallway.

28. **Spreading out people**

$$\text{Maximize } d = f(x_1,y_1,\ldots,x_n,y_n) = \min_{i<j} \sqrt{(x_i-x_j)^2+(y_i-y_j)^2}.$$

Square constraints: $0 \le x_i, y_i \le 1$.
Unit-disk constraints: $g_i(x_i,y_i) = x_i^2 + y_i^2 \le 1$.

29. **Shortest delivery route**

$$\text{Minimize } D = f(\sigma) = \sum_{k=1}^{n} d_{\sigma(k),\sigma(k+1)}, \qquad \sigma(n+1) = \sigma(1).$$

Constraint: σ is a permutation of all home labels.

30. **Fire stations**

$$\text{Minimize } D = f(S_1,S_2,S_3) = \max_i \min_{1\le j\le 3} \|H_i - S_j\|.$$

Alternatively,

$$\text{Minimize } D_{\text{avg}} = f(S_1,S_2,S_3) = \frac{1}{n}\sum_{i=1}^{n} \min_{1\le j\le 3} \|H_i - S_j\|.$$

Constraints: S_1, S_2, S_3 lie in the allowed city region.

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31. **Fastest roller coaster**

$$\text{Minimize } T = f(\gamma) = \int_\gamma \frac{ds}{\sqrt{2gy}}.$$

Constraints: γ joins the prescribed endpoints; y measures distance downward from the start, with $y > 0$ in the interior.

32. **Soap film**

$$\text{Minimize } A = f(S) = \iint_S dS.$$

Constraint: $\partial S = C_1 \cup C_2$, where C_1, C_2 are the prescribed circular rings. Require S to be connected if the film must connect the rings.

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