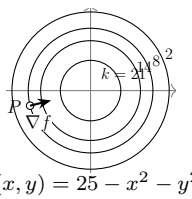
Chapter 14 Review

14.1	Domain: Region in xy-plane where $f(x,y)$ is defined. Be able to find and graph this region.	Level curves: 2D curves given by $f(x,y)=k$. Graph curves in xy-plane for several values of k.
14.3	$f_x = \dfrac{\partial z}{\partial x}$ = slope in x-direction $f_{xx} = \dfrac{\partial^2 z}{\partial x^2}$ = concavity in x-direction $f_{xy} = \dfrac{\partial^2 z}{\partial y \partial x}$ = mixed second partial	$f_y = \dfrac{\partial z}{\partial y}$ = slope in y-direction $f_{yy} = \dfrac{\partial^2 z}{\partial y^2}$ = concavity in y-direction $f_{xy} = f_{yx}$ (Clairaut's Theorem)
14.4	$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$ $dz = f_x(a,b)\,dx + f_y(a,b)\,dy$ $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ $\langle 1, 0, f_x(a,b)\rangle$ = tangent in x-direction $\langle 0, 1, f_y(a,b)\rangle$ = tangent in y-direction	Tangent plane at (a,b) Total differential (approximate change in z) $f(x,y) \approx L(x,y)$ near (a,b) $\mathbf{n} = \langle -f_x(a,b), -f_y(a,b), 1\rangle$ normal to surface/tangent plane If $F(x,y,z) = z - f(x,y)$, then $\mathbf{n} = \nabla F$.
14.5	Given $z = f(x,y)$, $x = x(s,t)$, $y = y(s,t)$ $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$ $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$	Given $x = x(t)$, $y = y(t)$ $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$ Implicit Shortcut: $F(x,y,z) = 0$ $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}, \quad \dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$
14.6	$\nabla f(x,y) = \langle f_x, f_y\rangle$ $\nabla f(x,y,z) = \langle f_x, f_y, f_z\rangle$ $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ = slope in unit direction $\mathbf{u} = \langle h, k\rangle$ $\langle h, k, D_{\mathbf{u}} f(a,b)\rangle$ = tangent vector in $\mathbf{u}$-direction	Max slope $= \lvert\nabla f\rvert$, direction ∇f Min slope $= -\lvert\nabla f\rvert$, direction $-\nabla f$ ∇f is perpendicular to level curves/surfaces of f.
14.7	Critical points: BOTH $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or one DNE (where f is defined). Global max/min: Evaluate f at candidates from inside (critical pts) and boundaries (critical numbers and endpts).	Second Derivative Test: For a critical point (a,b), $D(a,b) = f_{xx}f_{yy} - (f_{xy})^2$ $D > 0,\ f_{xx} > 0$: local min $D > 0,\ f_{xx} < 0$: local max $D < 0$: saddle point $D = 0$: inconclusive
14.8	2D: $f(x,y)$ subject to $g(x,y) = c$ $\nabla f = \lambda \nabla g$ solve together: (1) $f_x = \lambda g_x$, (2) $f_y = \lambda g_y$, (3) $g(x,y) = c$	3D: $f(x,y,z)$ subject to $g(x,y,z) = c$ $\nabla f = \lambda \nabla g$ solve together: (1) $f_x = \lambda g_x$ (2) $f_y = \lambda g_y$ (3) $f_z = \lambda g_z$ (4) $g(x,y,z) = c$

14.1/14.3/14.6/14.7 — Contour Maps Contours: $f(x,y) = k$ in the xy-plane.

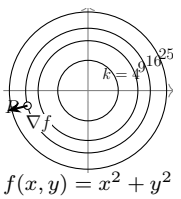
Local maximum



$f(x,y) = 25 - x^2 - y^2$

local max at $(0,0)$

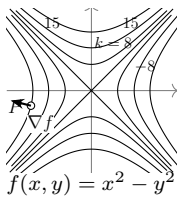
Local minimum



$f(x,y) = x^2 + y^2$

local min at $(0,0)$

Saddle point



$f(x,y) = x^2 - y^2$

saddle point at $(0,0)$

At the same point $P = (-4,-1)$: f_x is the slope moving right, f_y the slope moving up, and ∇f points uphill, perpendicular to the level curve.

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14.5 — Implicit Differentiation, Two Methods

Find $\dfrac{\partial z}{\partial x}$ at $(1,2,1)$ if $xz + z^2 + y^2 = 6$, where $z = f(x,y)$.

Classic method

Differentiate $xz + z^2 + y^2 = 6$ with respect to x:

$$z + x\frac{\partial z}{\partial x} + 2z\frac{\partial z}{\partial x} = 0$$

At $(1,2,1)$: $1 + 3\dfrac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \boxed{\dfrac{\partial z}{\partial x} = -\dfrac{1}{3}}$.

14.5 shortcut

$F(x,y,z) = xz + z^2 + y^2 - 6 = 0$

$F_x = z$, $F_z = x + 2z$, so

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{z}{x+2z} = \boxed{-\frac{1}{3}}.$$

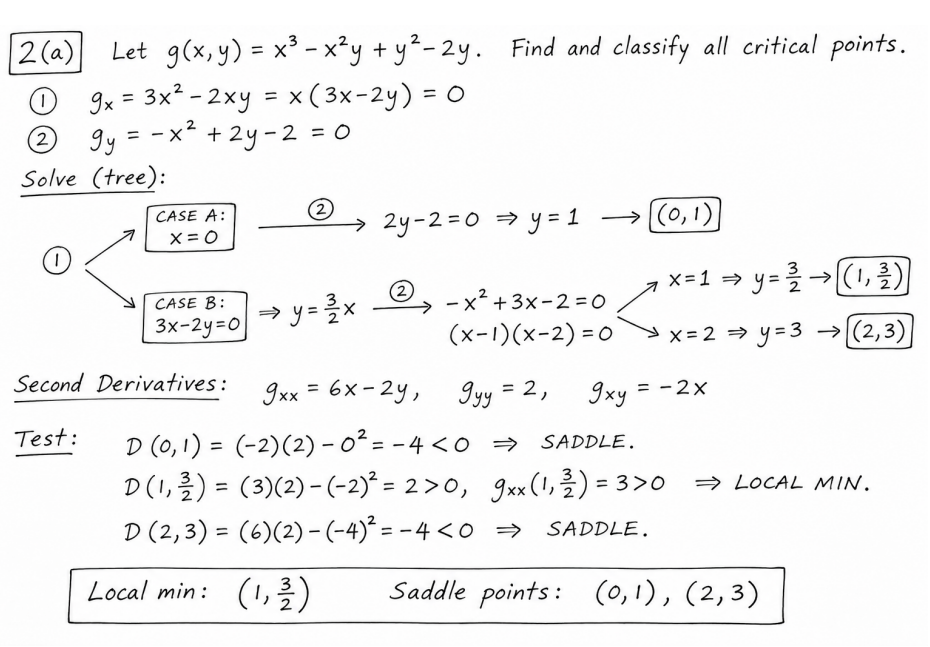
14.7 — Finding and Classifying Local Max/Min

1. Find f_x and f_y. (*verify* $f_{xy} = f_{yx}$.)
2. Set them BOTH equal to zero, then COMBINE (*try Loveless tree method*)
   - Pick one equation, factor first *if possible* to break into cases.
   - In each case solve for one variable, substitute into the other equation and solve.
   - This should give (x,y) points. (*verify* $f_x = f_y = 0$.)
3. Find f_{xx}, f_{yy}, and f_{xy}. At each critical point, compute $D = f_{xx}f_{yy} - (f_{xy})^2$ and use the second derivative test. (*think about concavity*)

Also check points where f_x or f_y does not exist (but f is defined).

Worked Example — Spring 2026 Exam 2, Q2(a)

Find and classify all critical points of $g(x,y) = x^3 - x^2y + y^2 - 2y$.



More practice: Spring 2024 Exam 2, Q2(a) (factor into cases); Spring 2013 Exam 2, Q2(b) (substitution).

14.7 — Finding Global (Absolute) Max/Min on a Closed, Bounded Region

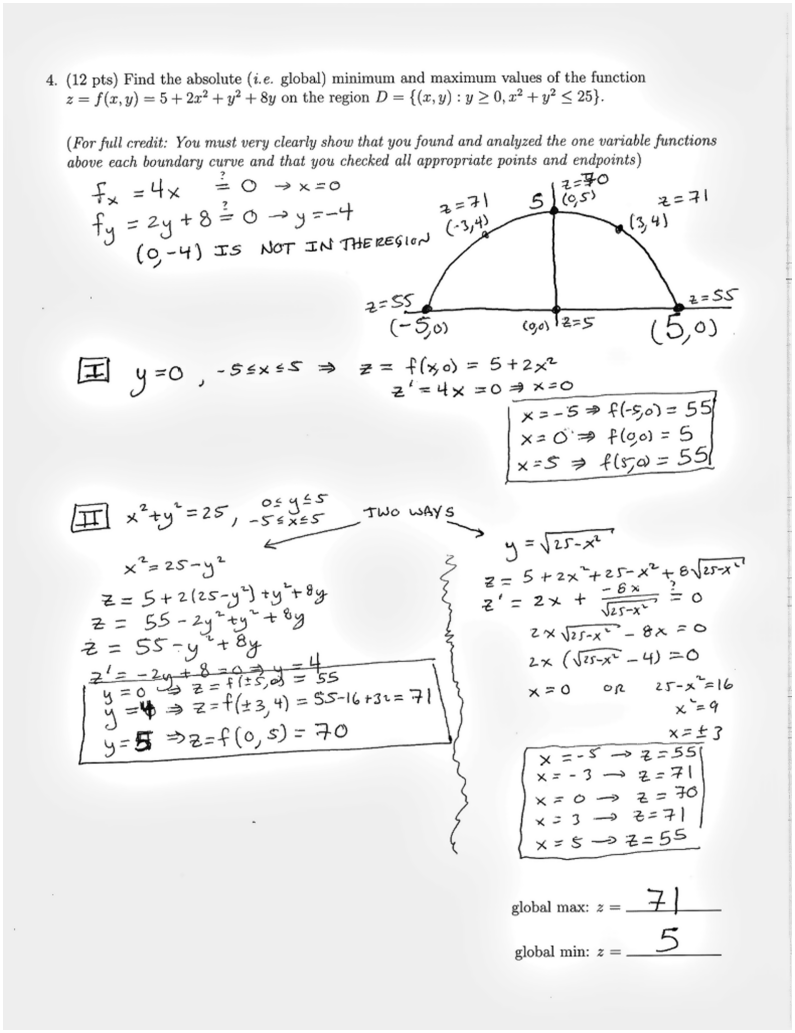
1. *Draw Region:* Label all boundaries, corners, and extreme x and y values.
2. *Interior:* Find all critical **points** inside the region (no 2nd derivative test needed). Only keep points properly inside the region.
3. *Boundaries:* For EACH boundary:
   - Substitute boundary equation into $f(x, y)$ to get a one-variable function. (*label each*)
   - Find critical **numbers** (derivative $= 0$ or DNE) within the boundary interval.
   - Include endpoints of boundary. (*note all* (x, y) *candidates*)

   Alternatively: Use Lagrange multipliers (14.8) on smooth boundaries. Still check endpoints.
4. *Compare:* Evaluate $f(x, y)$ at ALL candidates: interior critical points, boundary critical-number points, and boundary endpoints. The largest output is the absolute max and the smallest is the absolute min. (*show evaluations*)

Worked Example — Winter 2024 Exam 2, Q4

Find the absolute (global) minimum and maximum values of $z = f(x, y) = 5 + 2x^2 + y^2 + 8y$ on the region $D = \{(x, y) : y \geq 0,\ x^2 + y^2 \leq 25\}$.

Substitution



Lagrange Multipliers

Alternative: Lagrange on the semicircle boundary

Let $f(x,y) = 5 + 2x^2 + y^2 + 8y$, $g(x,y) = x^2 + y^2 = 25$.

$\nabla f = \langle 4x, 2y + 8 \rangle$, $\nabla g = \langle 2x, 2y \rangle$.

Set $\nabla f = \lambda \nabla g$:
$$\begin{cases} \text{①} & 4x = 2\lambda x \\ \text{②} & 2y + 8 = 2\lambda y \\ \text{③} & x^2 + y^2 = 25 \end{cases}$$

Case 1: $x = 0$. From ③: $y^2 = 25 \Rightarrow y = 5$ (since $y \geq 0$).
Point: $(0, 5)$.
$z = f(0, 5) = 5 + 0 + 25 + 40 = 70$.

Case 2: $x \neq 0$. From ①: $\lambda = 2$. Then from ②: $2y + 8 = 4y \Rightarrow y = 4$.
From ③: $x^2 + 4^2 = 25 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.
Points: $(-3, 4)$, $(3, 4)$.
$z = f(\pm 3, 4) = 5 + 2(9) + 16 + 32 = 71$.

Lagrange boundary candidates: $(0, 5)$, $(-3, 4)$, $(3, 4)$.

Note: Lagrange handles the smooth semicircle boundary $x^2 + y^2 = 25$, but we still must check the flat boundary $y = 0$ and the endpoints $(-5, 0)$ and $(5, 0)$.

14.8 — Lagrange Multipliers: Finding Max/Min Subject to a Constraint

1. *Identify:* Write down the objective function f and constraint $g = c$.
2. *Set Up:* Find ∇f and ∇g. Set $\nabla f = \lambda \nabla g$ and include $g = c$. (*label equations*)
3. *Solve:* COMBINE all equations and solve for the variables and λ.
   - Try factoring equations to break into cases. (*Loveless tree method*)
   - In each case use substitution or elimination to solve the remaining equations.
   - Write down all candidate points.
4. *Compare:* Evaluate f at all candidates to find the max/min over the constraint.

Big idea: Lagrange multipliers give us another way to analyze smooth boundaries without first substituting to eliminate a variable. They also let us directly solve applied problems with equality constraints. For global max/min over a region, still check interior points, other boundary pieces, and corners.

Basic Example: Find the absolute max/min of $f(x, y) = xy$, subject to $x^2 + y^2 = 8$.

1. *Identify:* $f(x, y) = xy$, $g(x, y) = x^2 + y^2 = 8$.
2. *Set up and label:*
$$\underbrace{y = 2\lambda x}_{(1)}, \quad \underbrace{x = 2\lambda y}_{(2)}, \quad \underbrace{x^2 + y^2 = 8}_{(3)}.$$
3. *Solve:* Since neither x nor y can be zero in (1)–(3), combining (1) and (2) gives $x^2 = y^2$.
$$\begin{aligned} &\text{Case 1: } y = x \quad \Rightarrow \; 2x^2 = 8 \Rightarrow (x, y) = (2, 2), (-2, -2), \\ &\text{Case 2: } y = -x \quad \Rightarrow \; 2x^2 = 8 \Rightarrow (x, y) = (2, -2), (-2, 2). \end{aligned}$$
4. *Compare:* $f(2, 2) = f(-2, -2) = 4$, while $f(2, -2) = f(-2, 2) = -4$.

Absolute maximum: 4. **Absolute minimum:** -4.

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14.7/14.8 — Applied Max/Min Problems

1. *VISUALIZE/LABEL:* Draw a good picture and label everything with variables.
2. *WHAT IS GIVEN?:* Write down all the given constraints.
3. *WHAT TO OPTIMIZE?:* Write down a formula for that quantity. Then, using the given facts, find a two (or three) variable function for the quantity that you want to optimize.
4. *DOMAIN?* Over what region does the problem make sense?
5. *USE CALCULUS:* Find the critical points and check the boundary. Use substitution or Lagrange multipliers when appropriate.
6. *JUSTIFY/VERIFY:* Make sure you actually did find a max or min as desired.

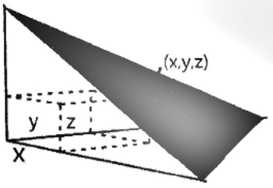
Worked Example — Fall 2018 Exam 2, Q4

Find the x, y, z dimensions of the rectangular box with maximum volume in the first octant with all vertices in the coordinate planes except one corner on the plane $4x + 3y + z = 12$. Clearly use the 2nd derivative test to verify your point gives a local max.

Substitution

4. (12 pts) Find the x, y, z dimensions of the rectangular box with maximum volume in the first octant with all vertices (corners) in the coordinate planes except one vertex (corner) that is on the plane $4x + 3y + z = 12$. (One example of such a rectangular box is shown)

At the end, ***clearly*** use the 2nd derivative test to verify your point gives a local max.



MAXIMIZE VOLUME OF BOX $= xyz$

SUBJECT TO CONSTRAINT $z = 12 - 4x - 3y$

$\text{VOL} = f(x,y) = 12xy - 4x^2y - 3xy^2$

$f_x = 12y - 8xy - 3y^2 \stackrel{?}{=} 0 \Rightarrow y(12 - 8x - 3y) \stackrel{?}{=} 0$

$f_y = 12x - 4x^2 - 6xy \stackrel{?}{=} 0 \quad 2x(6 - 2x - 3y) \stackrel{?}{=} 0$

$12 - 8x - 3y \stackrel{?}{=} 0 \Rightarrow y = \frac{12 - 8x}{3}$

COMBINE $6 - 2x - 3\left(\frac{12 - 8x}{3}\right) \stackrel{?}{=} 0$

$6 - 2x - 12 + 8x \stackrel{?}{=} 0 \Rightarrow 6x = 6 \Rightarrow \boxed{x = 1}$

$\Rightarrow \boxed{y = 4/3}$

$\Rightarrow \boxed{z = 12 - 4 - 4 = 4}$

$\boxed{(1, \tfrac{4}{3}, 4) = (x, y, z)}$

2ND DERIV. TEST

$f_{xx} = -8y$

$f_{yy} = -6x$

$f_{xy} = 12 - 8x - 6y$

$f_{xx}(1, 4/3) = -\frac{32}{3} < 0$

$f_{yy}(1, 4/3) = -6 < 0$

$f_{xy}(1, 4/3) = 12 - 8 - 8 = -4$

LOCAL MAX!!!

$D = \left(-\frac{32}{3}\right)(-6) - (-4)^2 = 64 - 16 = 48 > 0$

Lagrange Multipliers

Alternative: Lagrange Multipliers

Let $V(x,y,z) = xyz$ and $g(x,y,z) = 4x + 3y + z = 12$. Use $\nabla V = \lambda \nabla g$:

① $yz = 4\lambda$
② $xz = 3\lambda$
③ $xy = \lambda$
④ $4x + 3y + z = 12$

From ①/③: $z = 4x$; from ②/③: $z = 3y$. Hence $4x = 3y \Rightarrow y = \frac{4}{3}x$.

Substitute $y = \frac{4}{3}x$ and $z = 4x$ into ④:

$4x + 3\left(\frac{4}{3}x\right) + 4x = 12 \Rightarrow 12x = 12 \Rightarrow x = 1, \; y = \frac{4}{3}, \; z = 4.$

Critical point: $\boxed{(1, \frac{4}{3}, 4)}$

Justify/Verify: On the boundary of the first-octant feasible region, where $x = 0$ or $y = 0$ or $z = 0$, the volume is 0, so this interior positive critical point gives the absolute maximum volume.

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