Lagrange Multipliers

Math 126 • Section 14.8 • Constrained Optimization

The idea

To maximize or minimize f subject to $g = c$, look for points where the two gradients are parallel:

$$\boxed{\nabla f = \lambda \nabla g \qquad \text{and} \qquad g = c.}$$

Geometrically: a level curve/surface of f is tangent to the constraint at a regular constrained extremum.

The method

one constraint

1. Identify f (what to optimize) and $g = c$ (the constraint).

2. Find ∇f and ∇g; write $\nabla f = \lambda \nabla g$ **and** the constraint.

3. Solve the system for all possible points (x, y) or (x, y, z).

4. Plug each point into f; compare outputs to find the largest/smallest.

$$\textbf{Two variables:} \quad f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x,y) = c.$$

$$\textbf{Three variables:} \quad f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z, \quad g(x,y,z) = c.$$

Tips for solving the equations

Eliminate λ. Compare equations or substitute. If denominators are nonzero, you can use

$$\frac{f_x}{g_x} = \frac{f_y}{g_y} \quad \left(= \frac{f_z}{g_z} \text{ in 3D} \right).$$

Watch for lost solutions. If you divide by x, y, z, or another expression, first check when it is zero. If an equation factors, consider *every* case.

Example 1

two variables

Find the absolute max/min of $f(x, y) = xy$ subject to $x^2 + y^2 = 2$.

$$\underbrace{\langle y, x \rangle}_{\nabla f} = \lambda \underbrace{\langle 2x, 2y \rangle}_{\nabla g} \qquad \Longrightarrow \qquad \begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2 + y^2 = 2. \end{cases}$$

Here $x, y \neq 0$ (check the equations and constraint). Thus $y/x = x/y$, so $y^2 = x^2$:

Case	points	value of $f = xy$
$y = x$	$(1,1),\ (-1,-1)$	1 **MAX**
$y = -x$	$(1,-1),\ (-1,1)$	-1 **MIN**

Example 2

three variables

Find the absolute max/min of $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 = 3$.

$$\langle 1,1,1 \rangle = \lambda \langle 2x, 2y, 2z \rangle \quad \Longrightarrow \quad x = y = z, \quad 3x^2 = 3 \quad \Longrightarrow \quad x = \pm 1.$$

$$\boxed{f(1,1,1) = 3 \text{ (max)}} \qquad \boxed{f(-1,-1,-1) = -3 \text{ (min)}}$$

Remember: The method finds candidates, not automatically maxima/minima. Check exceptional points where $\nabla g = \mathbf{0}$ and any other boundaries or corners. On a closed, bounded constraint with f continuous, compare all candidates for absolute extrema.

Lagrange Multipliers: Worked Examples

Math 126 • Section 14.8 • Three algebra patterns

1. Factor and split into cases

do not divide by x

Find the absolute maximum and minimum of $f(x,y) = x^2y$ on $x^2 + y^2 = 3$.

$$\nabla f = \langle 2xy, x^2\rangle, \quad \nabla g = \langle 2x, 2y\rangle$$
$$2xy = 2\lambda x, \quad x^2 = 2\lambda y, \quad x^2 + y^2 = 3.$$

First equation: $2x(y - \lambda) = 0$. Thus **either** $x = 0$ **or** $\lambda = y$.

Case A: $x = 0 \;\Rightarrow y = \pm\sqrt{3}$ $\qquad\qquad \Rightarrow f = 0.$

Case B: $x \neq 0 \;\Rightarrow \lambda = y \Rightarrow x^2 = 2y^2$ $\qquad \Rightarrow 3y^2 = 3,\ y = \pm 1,\ x = \pm\sqrt{2}.$

Compare: $f(\pm\sqrt{2}, 1) = 2$ **(max)**, $f(\pm\sqrt{2}, -1) = -2$ **(min)**; at $(0, \pm\sqrt{3})$, $f = 0$.

2. Three variables

equal squares, not equal signs

Find the absolute maximum and minimum of $f(x,y,z) = xyz$ on $x^2 + y^2 + z^2 = 3$.

$$\langle yz, xz, xy\rangle = \lambda\langle 2x, 2y, 2z\rangle,$$
$$yz = 2\lambda x, \qquad xz = 2\lambda y, \qquad xy = 2\lambda z, \qquad x^2 + y^2 + z^2 = 3.$$

If any variable is zero: $f = 0$ (these cannot give the absolute max or min once we find ± 1).
Otherwise: multiply each equation by its corresponding variable:

$$xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2.$$

Since $xyz \neq 0$, also $\lambda \neq 0$. Hence $x^2 = y^2 = z^2$; with the constraint,

$$3x^2 = 3 \quad \Rightarrow \quad |x| = |y| = |z| = 1.$$

Compare signs: the eight points $(\pm 1, \pm 1, \pm 1)$ give $xyz = \pm 1$.

$$\boxed{\text{Max } = 1 \text{ (even number of minus signs)}} \qquad \boxed{\text{Min } = -1 \text{ (odd number)}}$$

3. Closest point

optimize distance squared

Find the point on the plane $x + 2y + 2z = 9$ closest to the origin.

$$f = x^2 + y^2 + z^2 \quad \text{(distance squared)}, \qquad g = x + 2y + 2z = 9.$$

$$\langle 2x, 2y, 2z\rangle = \lambda\langle 1, 2, 2\rangle \quad \Rightarrow \quad x = \tfrac{\lambda}{2},\ y = \lambda,\ z = \lambda.$$

Substitute into the constraint:

$$\tfrac{\lambda}{2} + 2\lambda + 2\lambda = 9 \quad \Rightarrow \quad \tfrac{9}{2}\lambda = 9 \quad \Rightarrow \quad \lambda = 2.$$

$$\boxed{\text{Closest point: } (1,2,2)} \qquad \boxed{\text{Distance: } \sqrt{1+4+4} = 3}$$

There is no farthest point: the plane extends indefinitely. The Lagrange equations give a candidate, but the domain decides whether absolute extrema exist.