A Small Bit of Physics

Vectors are a very useful way to study forces, acceleration, velocity, and position in physical problems. We will see a bit more of this as the quarter progresses, but time makes it difficult for us to go much beyond the basics. So here is a small and simplistic taste of why a scientist might like vectors.

Accurately modeling quantities: First off, several concepts in physics naturally have magnitude and direction so they are vectors. Here are some quantities that are often represented by vectors: Force, acceleration, velocity, momentum, torque. Other quantities are *scalars* (numbers) that don't have a specified direction including: mass, length, speed, energy, time, and temperature.

Vector Tug of War: If several forces are acting on an object say $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{F_3}$, then the resultant force on the object is given by

$$\text{Resultant Force } = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3}.$$

Some notable examples:

1. If an object is at rest (not moving), then the sum of the forces must be the zero vector (they all cancel out). (Aside: When we are talking about the force on a rope or a wire, we often use the word *tension*. In this setting tension is force.)

   - (A simple example) Two people are playing tug-o-war. One person is pulling east at a force of 30 N. If the rope is not moving, what is the force for the other person?
     ANSWER: The answer is the other person is pulling at 30 N toward the west. But let me say this in terms of vectors. If east is the positive x-axis, then we have the first person pulling with force vector $\langle 0, 30\rangle$. Say the other person is pulling with force $\langle v_1, v_2\rangle$. Since the rope is still, the sum of their forces is the zero vector, so we get $\langle v_1, v_2\rangle + \langle 0, 30\rangle = \langle 0, 0\rangle$. Add components and equating gives $v_1 + 0 = 0$ and $v_2 + 30 = 0$, so $\langle v_1, v_2\rangle = \langle 0, -30\rangle$ (no big suprise).
   - Three ropes are tied to the same ring and we play three way tug-o-war. One of the ropes is pulled with force vector $\langle 1, 0, 1\rangle$, another rope is pulled with force vector $\langle -3, 1, 0\rangle$. If the ring in the middle is not moving, what is the force vector for the third rope?
     ANSWER: Label the force on the third rope as $\langle v_1, v_2, v_3\rangle$. Since the rope is still, the sum of their forces is the zero vector, so we get $\langle v_1, v_2, v_3\rangle + \langle 1, 0, 1\rangle + \langle -3, 1, 0\rangle = \langle 0, 0, 0\rangle$. Add components and equating gives $v_1 - 2 = 0$, $v_2 + 1 = 0$, and $v_3 + 1 = 0$, so $\langle v_1, v_2, v_3\rangle = \langle 2, -1, -1\rangle$.
   - An object is on a 45 degree inclined ramp. The object is at rest, what is the frictional force?
     ANSWER: There are three forces acting on the object and they all must sum to the zero vector. The first force is gravity which is given by the vector $\langle 0, -mg\rangle$. The next force is the *normal force* which is the force of the ground pushing up on the object (it isn't falling through the incline). The normal force vector is the projection of the vector opposing gravity (*i.e.* $\langle 0, mg\rangle$) onto any normal vector to the incline, for example $\langle -1, 1\rangle$ in this case (if the ramp slopes down from right to left). Calculating we get the normal force vector then is $\text{proj}_{\langle -1,1\rangle}\langle 0, mg\rangle = \left\langle -\frac{1}{2}mg, \frac{1}{2}mg\right\rangle$. If the frictional force is $\langle v_1, v_2\rangle$, then we get $\langle v_1, v_2\rangle + \left\langle -\frac{1}{2}mg, \frac{1}{2}mg\right\rangle + \langle 0, -mg\rangle = \langle 0, 0\rangle$. Adding and equating components gives $v_1 - \frac{1}{2}mg + 0 = 0$ and $v_2 + \frac{1}{2}mg - mg = 0$, which gives $\langle v_1, v_2\rangle = \left\langle \frac{1}{2}mg, \frac{1}{2}mg\right\rangle$. The magnitude of this frictional force vector is $\frac{1}{\sqrt{2}}mg$.

2. If the object is not at rest (it is moving), then it is still true that the sum of the forces is the resultant force. We can use this with Newton's Second Law which says $\mathbf{F} = m\mathbf{a}$ (Force equals mass times acceleration). Hence, we get that acceleration satisfies $\mathbf{a} = \frac{1}{m}\mathbf{F}$, so we can start to describe the motion.

   - Assume a ball with mass 2 kg is dropped from a hot air balloon that is 300 meters high. There is a strong and steady wind from east to west that results in a force of magnitude 5 N on the ball. If the only other force on the ball is the force due to gravity, give the acceleration vector.
     ANSWER: In 3D, let's label the positive x-direction as west and the location of the balloon at (0,0,300). So the wind has force vector $\langle 5, 0, 0\rangle$. Gravity results in a force vector of $\langle 0, 0, -mg\rangle = \langle 0, 0, -2(9.8)\rangle = \langle 0, 0, -19.6\rangle$. Thus, the resultant force is $\langle 5, 0, -19.6\rangle$.
     By Newton's second law, $\mathbf{a}(t) = \frac{1}{m}\mathbf{F} = \frac{1}{2}\langle 5, 0, -19.6\rangle = \langle 2.5, 0, -9.8\rangle$.
     Going further:
     In Chapter 13, we will learn that we can find the antiderivatives component-wise to get the velocity and positive vectors which would give $\mathbf{v}(t) = \langle 2.5t + C_1, C_2, -9.8t + C_3\rangle = \langle 2.5t, 0, -9.8t\rangle$ (the initial velocity vector is zero, because the object was dropped, which is why all the constants of integration are zero). And integrating again gives
     $\mathbf{r}(t) = \left\langle 1.25t^2 + D_1, D_2, -4.9t^2 + D_3\right\rangle = \left\langle 1.25t^2, 0, -4.9t^2 + 300\right\rangle$ (which is the position function).

Work with vectors: From calculus 2, you know that for a constant force acting on an object over a given distance, we define WORK = (FORCE)(DISTANCE). You discussed in calculus what to do if force or distance wasn't constant and how to write it in terms of an integral. This discussion was all in terms of scalar quantities. In many situations, force and distance/displacement are given in terms of vectors. When a constant force vector, $\mathbf{F}$, is acting on an object that is displaced along a distance vector, $\mathbf{D}$, then the work is given by WORK = $\mathbf{F} \cdot \mathbf{D}$ (this is the dot product). Note that work is still a scalar quantity (typically measured in Joules or foot-pounds).

- (A simple example) You are pulling a wagon 200 feet along a horizontal surface. You are holding the handle at an angle of 30 degree with a force of 40 pounds. How much work is done?
  ANSWER: If we set up a coordinate system with the start as the origin and the direction we are going as the positive x-axis, we have that displacement/distance vector is $\mathbf{D} = \langle 0, 200 \rangle$. If we draw the force vector and make a triangle, we find that the components of the force vector satisfy $x = 40\cos(30) = 20\sqrt{3}$ and $y = 40\sin(30) = 20$. Thus, the force vector is $\mathbf{F} = \left\langle 20\sqrt{3}, 20 \right\rangle$. The component of force in the direction of the wagon is $40\cos(30) = |\mathbf{F}|\cos(30)$.

  So using the basic definition from calculus 2, we

  $$\text{Work } = 40\cos(30)200 = |\mathbf{F}|\cos(30)|\mathbf{D}| = |\mathbf{F}||\mathbf{D}|\cos(30)$$

  By the main fact from dot products we get

  $$\text{Work } = \mathbf{F} \cdot \mathbf{D} = (0)(20\sqrt{3}) + (200)(20) = 4000 \text{ ft-lbs.}$$

Torque: measures the tendency of an object to rotate. Torque is defined to be $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is the position vector and $\mathbf{F}$ is the force vector. If the wrench you are using is long, then, by leverage, you will need less force than if the wrench is short (Law of the Lever). So we'd like to define torque so that it's magnitude is the product of the length of the lever times the magnitude of the force in the direction perpendicular to the lever. That is, we want to define $|\boldsymbol{\tau}| = |\mathbf{r}|(\text{force in direction perpendicular to } \mathbf{r}) = |\mathbf{r}||\mathbf{F}|\sin(\theta)$ where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$. By one of the main cross product facts, $|\mathbf{r}||\mathbf{F}|\sin(\theta) = |\mathbf{r} \times \mathbf{F}|$. So this gives a motivation for the definition of torque as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.