

12.3 Review

In 12.3, we learn about dot products.

1. For two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, we define

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = \text{'the dot product of } \mathbf{a} \text{ and } \mathbf{b}'$$

2. Essential Properties (doing some examples I think you can quickly convince yourself that these are true, go ahead and try). You should recognize that these properties essentially tell us that the dot product satisfies all the standard rules that we are used to using for regular products (commutativity, distribution, etc):

(a) $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$.

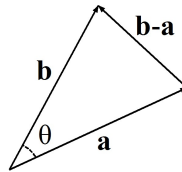
(b) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

(c) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(d) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$

(e) $c(\mathbf{a} \cdot \mathbf{b}) = (c\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{b})$

3. Aside (This is just for your information, you can skip this and go to fact 4 if you wish): The number one biggest fact for us that relates to dot products comes from the law of cosines which is a useful generalization of the Pythagorean theorem. It says for any triangle as below with θ the angle between \mathbf{a} and \mathbf{b} as shown



we have the following relationship between the sides $|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta)$ (notice if $\theta = \pi/2$ then you just get a right triangle and this becomes the Pythagorean theorem).

Now we can use the facts above to simplify the left-hand side: $|\mathbf{b} - \mathbf{a}|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2$. Then subtracting $|\mathbf{a}|^2 + |\mathbf{b}|^2$ from both sides give $-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos(\theta)$. Dividing by -2 gives the key result.

4. **This is the most important fact from this section:** The first thing you should think of when you think dot product in this course is the following relationship. If θ is the angle between two nonzero vectors \mathbf{a} and \mathbf{b} when drawn tail to tail (with $0 \leq \theta \leq \pi$), then we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta).$$

Thus, the dot product gives some indirect information about the angle between two vectors and we can use this relationship to solve for that angle.

5. Important consequences

- (a) **The following is one of the biggest consequence from the important fact:**

If \mathbf{a} and \mathbf{b} are perpendicular (orthogonal), then the angle between them is $\pi/2$ so we would have $\mathbf{a} \cdot \mathbf{b} = 0$.

- (b) If \mathbf{a} and \mathbf{b} are parallel and in the same direction, then the angle between this is 0 so we would have $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$.
- (c) If \mathbf{a} and \mathbf{b} are parallel and in the opposite direction, then the angle between this is π so we would have $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.
6. In applications, one of the biggest uses of dot products is in the study of projections. Assume you have two vectors \mathbf{a} and \mathbf{b} drawn tail to tail. Consider the line that goes through \mathbf{a} . Now go straight down from the tip of \mathbf{b} to the line that goes through \mathbf{a} . That is called projecting \mathbf{b} down onto \mathbf{a} .

Our textbook defines two terms about this projection:

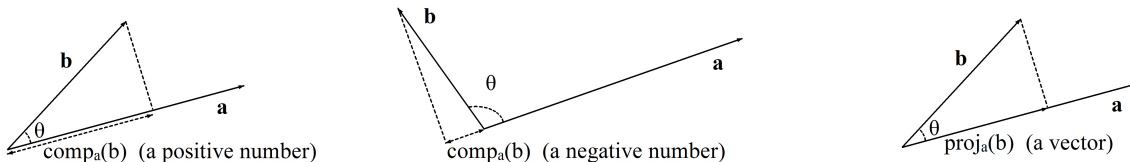
The scalar, or component, projection of \mathbf{b} onto \mathbf{a} is the length of the projection (it is a number that is either positive or negative depending on whether that length is in the same direction as \mathbf{a} or in the opposite direction of \mathbf{a}). We found in class that this number is given by

$$\text{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$

The vector projection of \mathbf{b} onto \mathbf{a} is the actual vector you get from the projection. So you just need to scale \mathbf{a} down to a unit vector, then multiply it by the number you get from the component projection to get the projection vector. As we saw in class:

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{1}{|\mathbf{a}|} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}.$$

These concepts are visualized below:



7. Projections give us one visual interpretation of dot products. Namely, if \mathbf{T} is a unit vector (meaning $|\mathbf{T}| = 1$) and you are given some other vector \mathbf{a} that you want to project onto \mathbf{T} , then the formula above gives that the scalar projection would be $\frac{\mathbf{T} \cdot \mathbf{a}}{|\mathbf{T}|} = \mathbf{T} \cdot \mathbf{a}$. So if you are working with unit vectors, then the dot product is the same as the length of the projection! This is one of the main reasons we like unit vectors so much in applications. If we want to project a force in the direction of a unit vector all we have to do to get the magnitude of the force in that direction is to take a dot product.
8. In physics, you often first encounter dot products when discussing the concept of work. Suppose an object moves from point A to point B while be acted on by the force \mathbf{F} . (see the picture of the wagon being pulled in section 12.3 for a nice visual of such a situation). We say that $\mathbf{D} = \overrightarrow{AB}$ is the displacement vector. Using all the facts above we can make several observations:

$$\text{comp}_{\mathbf{D}}(\mathbf{F}) = \frac{\mathbf{D} \cdot \mathbf{F}}{|\mathbf{D}|} = \text{'the magnitude of the force in the direction the object is moving'}$$

In Math 125, you saw the definition that work is defined to be 'force times distance'. So if you want to know the work done by this force over this displacement, then

$$\text{Work} = (\text{Magnitude of force in direction of } \mathbf{D}) \text{ times (length of } \mathbf{D}) = \left(\frac{\mathbf{F} \cdot \mathbf{D}}{|\mathbf{D}|} \right) |\mathbf{D}| = \mathbf{F} \cdot \mathbf{D}.$$